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RHEOLOGY OF A DILUTE SUSPENSION OF AXISYMMETRIC BROWNIAN PARTICLES

HOWARD BRENNER

Department of Chemical Engineering, Carnegie–Mellon University, Pittsburgh, PA. 15213, U.S.A.

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Abstract—Explicit results are presented for the complete rheological properties of dilute suspensions of rigid, axisymmetric Brownian particles possessing fore–aft symmetry, when suspended in a Newtonian liquid subjected to a general three-dimensional shearing flow, either steady or unsteady. It is demonstrated that these rheological properties can be expressed in terms of five fundamental material constants (exclusive of the solvent viscosity), which depend only upon the sizes and shapes of the suspended particles. Expressions are presented for these scalar constants for a number of solids of revolution, including spheroids, dumbbells of arbitrary aspect ratio and long slender bodies. These are employed to calculate rheological properties for a variety of different shear flows, including uniaxial and biaxial extensional flows, simple shear flows, and general two-dimensional shear flows. It is demonstrated that the rheological properties appropriate to a general two-dimensional shear flow can be deduced immediately from those for a simple shear flow. This observation greatly extends the utility of much of the prior Couette flow literature, especially the extensive numerical calculations of Scheraga *et al.* (1951, 1955).

The commonality of many disparate results dispersed and diffused in earlier publications is emphasized, and presented from a unified hydrodynamic viewpoint.

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1. INTRODUCTION

A very substantial body of literature exists pertaining to the rheological properties of dilute suspensions of rigid, neutrally buoyant, axisymmetric Brownian particles suspended in Newtonian liquids and subjected to homogeneous shearing flows, especially simple shear flows. (See, for example, the extensive reviews of Bird, Warner & Evans 1971 and Brenner 1972b.) Examples of bodies of revolution which have been studied in this manner are spheres (Einstein 1906, 1911), spheroids (Scheraga 1955, Giesekus 1962a, Brenner 1972a, Brenner & Condiff 1974, Leal & Hinch 1971, Hinch & Leal 1972, 1973), near spheres (Leal & Hinch 1972), "non-interacting" and "first-order" spherical dumbbells (Bird, Warner & Evans 1971, Bird & Warner 1971, Stewart & Sørensen 1972), spherical dumbbells of arbitrary aspect ratio (Wakiya 1971, Nir & Acrivos 1973), and long slender bodies possessing either pointed or blunt ends (Okagawa, Cox & Mason 1973). For the latter two classes of bodies rheological calculations have only been performed for situations in which the rotary Brownian movement is supposed negligible, and then only for the case of simple shear flows. Neglect of rotary diffusion leads to an indeterminacy in the rheological calculations of the type originally encountered by Jeffery (1922) in connection with spheroidal particles. This indeterminacy stems from the lack of a unique, time-independent distribution of particle orientations in such circumstances.

In general, apart from the work of Cox & Brenner (1971)—which does not explicitly include the rotary Brownian movement—the pertinent rheological theory has been developed anew for each different particle shape, and for each different type of homogeneous shear, e.g. simple shear and extensional flows. Moreover, even when both of these characteristics were fixed, it was generally left to different investigators to separately investigate the asymptotic behavior in the limits of both small and large rotary Péclet numbers (i.e. dominant and weak rotary Brownian movement). This has produced a diffuse and unwieldy body of literature on the subject, especially when one considers that those investigations which have been based upon energy dissipation methods (in contrast to dynamical methods) generally fail to yield complete rheological information in regard to such items as normal stresses. The difficulties of conceptually organizing all this information in a coherent manner are even further compounded by the inclusion of *unsteady* (homogeneous) flows in the class of fluid motions of rheological interest.

The only serious (and successful) attempt to date to present these rheological results within a unified conceptual framework is that of Bird *et al.* (1971) and Armstrong & Bird (1973), who limit themselves primarily to "non-interacting" dumbbells. (Indeed, they go even further by including non-rigid dumbbells, as for example in the case where the spheres comprising the dumbbell are connected by a Hookean spring.) Unfortunately, the "non-interacting" dumbbell constitutes a very special, indeed sometimes singular, case of a body of revolution, so that the quantitative connection between their dumbbell results and the analogous body of literature on spheroids is not evident. In particular, their analysis fails to stress the fundamental hydrodynamic theme common to all these problems, including non-axisymmetric bodies of any shape whatsoever.

In this paper we furnish a general dynamical rheological theory for axially symmetric particles (possessing fore–aft symmetry) which subsumes within its purview all prior rigid-body results in the literature as special cases. In particular, it is demonstrated that the rheological properties of dilute suspensions of such bodies, including the rotary Brownian diffusion, can be expressed in terms of the volume fraction of suspended particles, the viscosity of the homogeneous Newtonian carrier fluid, and five nondimensional scalar material constants which depend only upon the shape of the suspended particles. This conclusion applies to any type of homogeneous shear flow, either steady or unsteady.

These five fundamental material constants are purely hydrodynamic in origin, and may be derived from the solution of the quasistatic Stokes equations for a single translating-rotating axisymmetric particle of requisite shape suspended in a simple shear flow. To calculate the values of these constants it suffices to consider only the two special cases where the symmetry axis of the particle lies parallel and perpendicular, respectively, to the streamlines of a simple shear flow, the symmetry axis being perpendicular to the vorticity vector in both cases. Though these fundamental hydrodynamic constants are derived from the solution for a steady simple shear flow in the absence of rotary diffusion, the constants thus obtained are sufficient to calculate rheological properties for any homogeneous shear flow, simple or not, and steady or not, including the case where the rotary Brownian motion is sensible. The Stokes–Einstein equations (Brenner 1967) furnish the necessary link between low Reynolds number hydrodynamics and rotary diffusion.

Though attention is confined to axisymmetric particles, the manner in which the subsequent theory may be applied to any centrally-symmetric particles, or indeed particles of arbitrary shape, will be reasonably self evident.

The theory derived here is used to weave together disparate and fragmentary results dispersed in the prior literature, presenting these known results in a more general context, and deriving several new results along the way—especially for the cases of dumbbells of arbitrary aspect ratio and long slender bodies. In particular, it is pointed out that, contrary to what is commonly assumed, circular cylindrical rods are not adequately modelled by long thin prolate spheroids. Whether the ends of the particle are "pointed" (as in the case of a prolate spheroid) or "blunt" (as in the case of a cylindrical rod) proves crucial in relating rheological and analogous transport properties to the longitudinal and transverse dimensions of the long slender body.

Present results are also relevant to theories of streaming birefringence, since two of the five "rheological" parameters (B and D_r) are identical to those appearing in birefringence theories of the Peterlin & Stuart (1939a, 1939b) type. Moreover, the same orientational distribution function is common to both phenomena.

The remainder of this Introductory section is devoted to a summary of the essential contents of the present paper.

In Section 2 expressions are written down for the hydrodynamic force, torque and "stresslet" exerted by an incompressible Newtonian fluid upon an isolated, translating-rotating, rigid particle of arbitrary shape suspended in a general homogeneous shearing flow which extends to infinity. Rotational Brownian motion is not considered. These three dynamical parameters are linear functions of the viscosity of the carrier fluid, and of the translational and rotational slip velocities between particle and fluid, as well as of the undisturbed rate of strain. The proportionality coefficients in these linear relationships are second, third and fourth rank material tensors (and pseudotensors), dependent solely upon the geometrical configuration of the wetted particle surface; that is, upon the size and shape of the body. The general forms adopted by these material tensors for an axially symmetric particle possessing fore–aft symmetry (i.e. a center of symmetry) are deduced by geometric symmetry arguments, and expressed in terms of a body-fixed unit vector e_i drawn along the axis of revolution of the body. The proportionality coefficients appearing in these expressions, relating the material tensors to various linear combinations of the tensors $e_i e_j \cdots e_k e_l$ of appropriate tensorial rank, are scalars. These material scalars are dependent only upon the size and shape of the axisymmetric body.

Of these material scalars arising in the expressions for the force, torque and stresslet exerted on a body of revolution, only eight are independent. Of these, two are irrelevant to the rheological theory, which pertains to force-free particles, and a third is irrelevant in consequence of the fact that rotation of the body about its symmetry axis does not alter the orientation of this axis, thereby constituting a "dead" degree of freedom. Hence, the intrinsic rheological properties of a dilute suspension of identical, hydrodynamically non-interacting, force-free, rigid, axisymmetric, Brownian particles are ultimately determined by the remaining five material constants. One of these plays a dual role, in that it also enters as a hydrodynamic resistance coefficient in a Stokes–Einstein equation for the rotary diffusion coefficient. Hence, in this role, it appears as a parameter in the partial differential equation governing the orientational distribution function.

Various inequalities imposed upon these material constants by the positive–definite nature of the mechanical energy dissipation are derived in Section 2.

Explicit expressions for these five material constants (as well as certain auxiliary constants derived from these) are obtained in Section 3 for various axisymmetric bodies possessing fore–aft symmetry, including spheroids (with spheres, circular disks, and long thin prolate spheroids as special cases), long slender bodies possessing either pointed or blunt ends, and spherical dumbbells of arbitrary aspect ratio, including the limiting case where the spheres touch. Limiting expressions are also obtained for "non-interacting" dumbbells (composed of rigidly-connected spheres situated so far apart as to asymptotically satisfy the condition of no hydrodynamic interaction between them), as well as for "first-order" dumbbells (in which hydrodynamic interactions among the two spheres composing the dumbbell are taken account of, to terms of first order in the ratio of sphere radii to center-to-center separation distance). Dumbbells of these latter types are of special interest in connection with the rheological properties of "stiff" macromolecular chains (Bird *et al.* 1971).

Material constants for the long slender bodies and dumbbells are extracted from the respective analyses of Cox (1970, 1971) and Okagawa *et al.* (1973), and of Wakiya (1971), each of which pertains only to the case of a steady simple shearing motion in the absence of rotary diffusion. Despite the restricted nature of the special class of problems from which these coefficients were extracted, the constants themselves suffice to analyze much more general rheological problems, including arbitrary homogeneous shearing flows (steady or unsteady) in circumstances where rotary diffusion is sensible. Such is the advantage of the broader context advocated in the present paper.

Section 4 provides a general theory of the rheological properties of suspensions of the type under consideration, correct to terms of the first order in the volume fraction of suspended particles. In particular, an expression is derived relating the mean deviatoric stress in the suspension to the mean velocity gradient (or, equivalently, the mean rate of strain tensor and mean vorticity vector) for steady, homogeneous shearing flows. In general, this relation (cf. [4.27]) is highly nonlinear. Utilization of this theory requires knowledge of the five fundamental material constants, as well as the second moment of the orientational distribution function. The latter can be calculated by solving the second-order partial differential equation governing this distribution function. For a prescribed mean velocity gradient, the only parameters appearing in the latter equation are two of the above five material constants. Hence, no new material constants are required to parametrize the second moment. In turn, this leads to the conclusion that the original five material constants suffice for a complete rheological theory.

The remainder of the paper is devoted to various applications of the general theory outlined in Section 4 to various important classes of shearing flows.

Calculations are presented in Section 5 of the explicit rheological properties arising during axisymmetric (i.e. uniaxial) extensional and compressive flows. It is demonstrated that the rheological behavior encountered during such flows can be represented by a single scalar viscosity coefficient, which is a function of the fractional rate of elongation G, all other things being equal. Through use of an exact expression for the dependence of the orientational distribution function upon G, an expression is derived for the variation of the intrinsic viscosity function with G, represented in dimensionless form by a rotary Péclet number $P = G/D_r$. Asymptotic values of this intrinsic viscosity, derived from the exact solution, are given for both small and large P. Graphs depicting the variation of intrinsic viscosity with dimensionless deformation rate are given for both oblate and prolate spheroids of various aspect ratios over the entire Péclet number range, $-\infty < P < \infty$.

In agreement with the numerical calculations of Clarke (1973) for this case, it is found that the rheological behavior is of the shear-thickening type for oblate spheroids over the complete range of shear rates, $-\infty < G < \infty$. However, for prolate spheroids this behavior obtains only for compressional flows. For prolate spheroids in extensional flows this behavior changes from shear thickening to shear thinning beyond a certain dimensionless shear rate (which depends upon the axis ratio of the spheroid) for circumstances in which the axis ratio exceeds 10.473.

Section 6 is devoted to a comparable rheological study for plane (i.e. two-dimensional, biaxial) extensional flows. In contrast with the uniaxial case of Section 5, rheological properties are no longer completely described by a single, shear-dependent viscosity function. Rather, two "normal" stress functions are now required. An exact expression is obtained for the dependence of the orientational distribution function upon the (dimensionless) rate of extension, and this is employed to derive explicit expressions for the variation of the intrinsic normal stresses with rotary Péclet number over the complete range of Péclet numbers. Results are presented graphically for both oblate and prolate spheroids, and limiting asymptotic expressions are derived for both small and large extensional rates.

Rheological properties are derived in Section 7 for general homogeneous shear flows, but only for small dimensionless shear rates (i.e. small rotary Péclet numbers). This restriction in the range of applicability comes about through the inability to obtain an exact solution for the orientational distribution function valid for all Péclet numbers. The (time-independent) Jaumann derivative, which expresses the proper material frame-indifference of the rheological constitutive equation when account is taken of material rotation, appears explicitly in the general constitutive relation for the mean deviatoric stresses. The appearance of this derivative stems directly from its original appearance in the expression for the orientational distribution function, which must itself manifest such invariance. That is, the distribution of particle orientations must be measured relative to the rotating material, if it is to possess physical significance. It is demonstrated, *inter alia*, that in the limit of zero Péclet number, where the disorienting effect of the rotary diffusion dominates over the orienting effect of the hydrodynamic stresses on the suspended particles, the rheological behavior is Newtonian. This corresponds to the case where the distribution of particle orientations is isotropic. Non-Newtonian behavior in rigid-particle suspensions generally arises from the anisotropic distribution of particle orientations engendered by the shear and vorticity fields.

Special attention is devoted in Sections 8–10 to the case of simple shear flows, the latter two sections being reserved for a discussion of the limiting behavior at large rotary Péclet numbers. In general, rheological behavior in simple shear flow can be expressed in terms of three viscometric functions—a viscosity function and two normal stress functions—each of which is generally shear-rate dependent. General expressions are derived for these viscometric functions in terms of material constants and three goniometric functions, each of which depends only upon the second moment of the orientational distribution function. These goniometric factors, which are fundamental to the theory of simple shear flow, depend upon the rotary Péclet number P and the dimensionless parameter B (derivable from the five basic material constants).

Analytic expressions which apply for $BP \ll 1$ are deduced for these goniometric factors by specializing the general results of Section 7. These, in turn, are utilized to obtain expressions for the three viscometric functions, valid for the case of small Péclet numbers. Where possible, these results are compared with prior results in the literature for spheroids, and "non-interacting" as well as "first-order" dumbbells.

Special attention is devoted to the limiting case where $B = 1$, which arises in the case of "non-interacting" dumbbells, as well as for other long-thin bodies of large aspect ratio. Here, by adaptation of the rheological results reported by Stewart & Sørensen (1972) for dumbbells, it proves possible to extract the goniometrical factors (appropriate to the value $B = 1$) over the complete range of Péclet numbers, $0 \leq P < \infty$. These goniometrical factors may then be applied to suspended particles other than dumbbells. It is pointed out that use of the "first-order" dumbbell theory of Bird & Warner (1971) and Stewart & Sørensen (1972) may result in appreciable errors in rheological calculations pertaining to such dumbbells, especially at large Péclet numbers.

By means of a simple transformation, the numerical values for the goniometrical factors derived from the Stewart–Sørensen work for $B = 1$ may also be utilized for the case where $B = -1$. These results then lead a calculation of the rheological properties of a suspension of circular disks in a simple shear flow over the complete range of Péclet numbers.

The numerical calculations of Scheraga *et al.* (1951, 1955), relating to the viscosity and streaming birefringence functions for prolate and oblate spheroids, are inverted so as to obtain the requisite goniometric factors as a function of B (or, equivalently, the "equivalent axis ratio" r_e) and of the Péclet number in the ranges $-1 < B < 1$ and $0 \leq P \leq 60$. Though derived from results pertaining to spheroids, these goniometric factors apply to any axisymmetric particle. In the limit where $B \to 1$ these three quantities show excellent agreement with the comparable numerical values derived from the Stewart & Sørensen (1972) analysis. These goniometric factors are employed to compute the normal stress functions for suspensions of prolate and oblate spheroids subjected to a simple shear flow, the viscosity function for such bodies already being available from the work of Scheraga (1955). These are compared with analytical asymptotic results derived for both small and large P. The primary and secondary normal stresses are of opposite algebraic sign, each increasing in magnitude from zero at zero shear rate (i.e. $P = 0$) to a finite upper limit at infinite shear rate ($P = \infty$).

The Scheraga tabulation of goniometric factors in Section 8 is limited to values of $P < 60$ in consequence of the rather slow numerical convergence of the doubly-infinite series used in their computation for values of P greatly in excess of 60. With this limitation in mind, Leal & Hinch (1971) and Hinch & Leal (1972) in a series of papers developed asymptotic solutions for spheroids, valid for the case where $P \gg 1$. Since spheroids possess the property that $|B| < 1$, this same restriction applies to the Leal–Hinch analysis. Section 9 is essentially a recapitulation of the Leal–Hinch theory, but adapted to axisymmetric particles of any shape. (The comparable problem for $|B| > 1$ is considered in Section 10.)

Two possible situations arise according as $P \gg r_e^3 + r_e^{-3}$ ("weak" Brownian motion) or $r_e^3 + r_e^{-3} \gg P \gg 1$ ("intermediate" case), wherein $r_e = [(B + 1)/(B - 1)]^{1/2}$. Numerical values of the goniometric factors for the "weak" case are shown to be in quite good agreement with the results of Scheraga *et al.* in the common region of overlap, thereby strengthening confidence in the numerical credibility of both sets of computations. In the "intermediate" case, large uncertainties exist in the Hinch & Leal (1972) numerical coefficients entering into the calculation of the goniometric factors. Here, the calculations of Stewart & Sørensen (1972) were employed to obtain presumably more accurate values for these coefficients than those given in the original Hinch & Leal (1972) paper. The goniometric factors for the "intermediate" case obtained in this manner show modestly good agreement with the numerical calculations of Scheraga in their common domain of validity.

The results of these large Péclet number asymptotic expansions are employed to obtain expressions for the viscometric functions appropriate to several different body shapes. A minor error in the Hinch & Leal (1972) expressions for the primary normal stress differences for spheroids is corrected.

The asymptotic analysis for $P \gg 1$ appropriate to bodies for which $|B| > 1$ is vastly different from that which obtains for $|B| < 1$. In particular, in the absence of rotary diffusion ($P = \infty$), and when $|B| > 1$, an axisymmetric body ultimately adopts a unique terminal orientation (relative to the principal axes of shear) which is independent of its initial orientation. In contrast, when rotary diffusion is absent, and when $|B| < 1$, the body undergoes a periodic rotation of the type first encountered by Jeffery (1922) for spheroidal particles. Whereas the limiting process $P \to \infty$ is singular for $|B| < 1$ it is uniform for $|B| > 1$. Calculation of the goniometric factors for $P \gg 1$ and $|B| > 1$ is brought to fruition in Section 10 by use of a method outlined in Hinch's (1971) thesis, after correcting an error in the latter's work. In effect, the orientational distribution function is Gaussian about the direction of the terminal orientation which obtains when the Brownian motion is wholly absent.

It is pointed out in Section 11 that the distribution function and concomitant moments thereof, required to compute the rheological properties for any two-dimensional flow (other than an irrotational flow), can be deduced directly from those for a simple shear flow by an appropriate re-interpretation of the physical significance of the rotary parameter B and shear rate G. Thus, the simple shear flow results of Sections 8–10 can be immediately adapted to calculate the rheological properties which obtain in almost every two-dimensional flow.

Reference is also made in Section 11 to Wayland's (1960) analysis of streaming birefringence in dilute suspensions subjected to arbitrary two-dimensional flows. The same goniometric factors required to calculate rheological properties appear in the streaming birefringence problem too. Rather than referring the distribution function to a system of material axes, Wayland chooses instead a set of rotating axes which, while translating with the fluid, maintain a fixed orientation relative to the local direction of the (generally curved) streamlines. It is demonstrated that the distribution function and, correspondingly, the requisite goniometric factors relative to such "intrinsic" axes are directly calculable in terms of comparable functions already available for a simple shear flow. This is done by means of an appropriate re-definition of the significance to be attached to the rotary parameter B and shear rate G. Once again, then, the detailed results of Sections 8–10 are shown to be adaptable to the solution of a rather more general class of two-dimensional flow problems than the simple shear for which they were originally derived.

Section 12 furnishes a general analysis of the rheological properties of a dilute suspension of axisymmetric Brownian particles subjected to an arbitrary unsteady flow. In particular, a general relation is derived expressing the deviatoric stress in terms of the time-dependent shear and vorticity tensors, and the second moment of the orientational distribution function. The same five material constants required for the rheological characterization of steady flows serve to uniquely characterize these unsteady flows too. The time-dependent Jaumann time derivative, which expresses the proper material frame-indifference of the rheological constitutive relation under rotation of the reference frame, arises naturally in the theory, without *a priori* anticipation of its appearance. By way of example, a detailed solution is given for the stress relaxation following the abrupt cessation of an arbitrary, steady flow. It is demonstrated, at least for the case where the prior steady flow is a simple shear, and for the case where $P \ll 1$, that the analysis yields results for the stress in agreement with the detailed calculations of Bird *et al.* (1971) for idealized dumbbells composed of non-interacting spheres.

2. MATERIAL TENSORS

Consider a single solid particle of arbitrary shape undergoing translation and rotation in an unbounded incompressible Newtonian fluid subject to a homogeneous shear $\mathbf{v}^{\infty}$ at infinity. Denote by O an arbitrary origin fixed in the particle. Let $\mathbf{U}$ be the translational velocity of this point and $\mathbf{\Omega}$ the angular velocity of the particle.

The undisturbed homogeneous shearing flow at infinity may be characterized by the constant velocity gradient tensor,

$$g_{ij} = v^{\infty}_{j,i}, \tag{2.1}$$

of which the symmetric and antisymmetric parts may be represented, respectively, by the symmetric rate of shear tensor,

$$s_{ij} = s_{ji} = \tfrac{1}{2}(g_{ij} + g_{ji}), \tag{2.2}$$

and the antisymmetric vorticity tensor

$$\lambda_{ij} = -\lambda_{ji} = \tfrac{1}{2}(g_{ji} - g_{ij}), \tag{2.3}$$

or, alternatively, the vorticity vector

$$\omega_i = \tfrac{1}{2}\varepsilon_{ijk}\lambda_{kj}, \tag{2.4}$$

with ε_{ijk} the permutation symbol. In terms of these, the undisturbed shear flow may be expressed in the form

$$v^{\infty}_i = v^o_i + \varepsilon_{ijk}\omega_j x_k + s_{ij}x_j, \tag{2.5}$$

wherein $\mathbf{v}^o$ denotes the value of the undisturbed shear flow at the point in the fluid presently occupied by the O, $\mathbf{x}$ being the position vector measured relative to this point.

All relevant Reynolds numbers based on particle size, translational and rotational velocities, and shear rate, are assumed sufficiently small compared with unity to justify neglecting nonlinear terms in the Navier–Stokes equations. Thus, the fluid motion $(\mathbf{v}, p)$ in the presence of the suspended particle is supposed governed by the quasistatic creeping motion equations

$$\nabla^2\mathbf{v} = \mu_o^{-1}\nabla_p, \quad \nabla\cdot\mathbf{v} = 0, \tag{2.6a, b}$$

in which μ_o is the viscosity of the homogeneous fluid. The no-slip boundary condition leads to the requirement that

$$v_i = U_i + \varepsilon_{ijk}\Omega_j x_k \tag{2.7}$$

at the particle surface. Moreover, at large distances from the particle,

$$v_i \sim v^{\infty}_i \quad \text{as } |\mathbf{x}| \to \infty, \tag{2.8}$$

corresponding to the requirement that the disturbance due to the presence of the particle in the shearing flow be attenuated at infinity.

In consequence of the linearity of the differential equations and boundary conditions, the hydrodynamic force $\mathbf{F}$, torque $\mathbf{L}$ (about O), and stresslet* $\mathbf{A}$, exerted by the fluid on the particle are linear vector functions of the translational slip velocity $\mathbf{U} - \mathbf{v}^o$, the rotational

* This stresslet is related to Batchelor's (1970) stresslet $\mathbf{S}'$ via the relation $A_{ij} = S'_{ij}/5\mu_0 V_p$.

slip velocity $\mathbf{\Omega} - \boldsymbol{\omega}$, and the shear $\mathbf{s}$. In the notation of Brenner (1972a) we therefore have that

$$F_i = \mu_o[{}^t\hat{K}_{ij}(v_j^o - U_j) + {}^c\hat{K}_{ji}(\omega_j - \Omega_j) + \hat{\Phi}_{ijk}s_{jk}], \quad [2.9]$$

$$L_i = \mu_o[{}^c\hat{K}_{ij}(v_j^o - U_j) + {}^r\hat{K}_{ij}(\omega_j - \Omega_j) + \hat{\tau}_{ijk}s_{jk}], \quad [2.10]$$

$$A_{ij} = M_{ijk}(v_k^o - U_k) + N_{ijk}(\omega_k - \Omega_k) + Q_{ijkl}s_{kl}. \quad [2.11]$$

The material tensors ${}^t\hat{K}_{ij}, \ldots, Q_{ijkl}$ in these relations are intrinsic properties of the particle alone, being dependent only upon its size and shape, i.e. only upon the geometric configuration of its wetted surface. These particle material tensors are constants relative to body-fixed axes, locked into the particle, which translate and rotate with the body. Following Brenner (1972a), in place of the careted material tensors in the expression for the torque, it proves convenient to introduce the related tensors

$${}^cK_{ij} = {}^c\hat{K}_{ij}/6V_p,\ {}^rK_{ij} = {}^r\hat{K}_{ij}/6V_p,\ \tau_{ijk} = \hat{\tau}_{ijk}/6V_p, \quad [2.12\text{a, b, c}]$$

where V_p denotes the particle volume.

In consequence of the symmetry relation $s_{ij} = s_{ji}$, one may arbitrarily set

$$\hat{\Phi}_{ijk} = \hat{\Phi}_{ikj},\ \tau_{ijk} = \tau_{ikj},\ Q_{ijkl} = Q_{ijlk}, \quad [2.13\text{a, b, c}]$$

leading to an appropriate reduction in the number of independent components of these tensors. Further reduction in this number occurs as a result of the incompressibility condition $s_{ii} = 0$, but we shall not pursue the consequences of such details here. Moreover, in view of the relations $A_{ij} = A_{ji}$ and $A_{ii} = 0$ (Brenner 1972a) we may write without loss of generality that

$$M_{ijk} = M_{jik},\ N_{ijk} = N_{jik},\ Q_{ijkl} = Q_{jikl}, \quad [2.14\text{a, b, c}]$$

and

$$M_{iik} = 0,\ N_{iik} = 0,\ Q_{iikl}s_{kl} = 0. \quad [2.15\text{a, b, c}]$$

The summation convention on repeated indices is utilized throughout.

In addition to these "trivial" symmetry relations, we have also the "kinetic" symmetry relations (Brenner 1964b)

$${}^t\hat{K}_{ij} = {}^t\hat{K}_{ji},\ {}^rK_{ij} = {}^rK_{ji}, \quad [2.16\text{a, b}]$$

and (Hinch 1972)

$$5V_pN_{ijk} = \hat{\tau}_{kij},\ 5V_pM_{ijk} = \hat{\Phi}_{kij},\ Q_{ijkl} = Q_{klij}, \quad [2.17\text{a, b, c}]$$

the first of which is equivalent to

$$N_{ijk} = \tfrac{6}{5}\tau_{kij}. \quad [2.18]$$

Bodies possessing a center of symmetry possess the property that (Brenner 1964b, 1964c)

$${}^c\hat{K}_{ij} = 0,\ \hat{\Phi}_{ijk} = 0,\ M_{ijk} = 0, \quad [2.19\text{a, b, c}]$$

provided that the origin O is chosen to lie at the center of symmetry. For such bodies, further reduction occurs in the number of independent components of the five remaining material tensors, but again we shall not enter into such details in the general case. Of special interest, however, is the case of a body of revolution possessing fore-aft symmetry (i.e. a plane of reflection symmetry normal to the symmetry axis). Such a body possesses a center of symmetry, which we will designate as the origin O.

Denote by $\mathbf{e}$ a body-fixed unit vector drawn along the symmetry axis of such an axisymmetric body, and let (e_1, e_2, e_3) be the components of this vector in any system of rectangular Cartesian axes (x_1, x_2, x_3), space-fixed or body-fixed. Then the forms adopted by the nonzero material tensors for such a particle are as follows:*

$$ {}^t\hat{K}_{ij} = e_i e_j {}^t\hat{K}_{\parallel} + (\delta_{ij} - e_i e_j){}^t\hat{K}_{\perp}, \tag{2.20} $$

$$ {}^rK_{ij} = e_i e_j {}^rK_{\parallel} + (\delta_{ij} - e_i e_j){}^rK_{\perp}, \tag{2.21} $$

$$ N_{ijk} = (\varepsilon_{ljk} e_l e_i + \varepsilon_{lik} e_l e_j)N, \tag{2.22} $$

$$ \tau_{ijk} = -(\varepsilon_{ijl} e_l e_k + \varepsilon_{ikl} e_l e_j)\tau, \tag{2.23} $$

$$ \begin{aligned} Q_{ijkl} = {} & (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})Q_1 + (\delta_{ij} e_k e_l + \delta_{kl} e_i e_j - 3e_i e_j e_k e_l)Q_2 \\ & + (\delta_{jk} e_i e_l + \delta_{il} e_j e_k + \delta_{ik} e_j e_l + \delta_{jl} e_i e_k - 4e_i e_j e_k e_l)Q_3. \end{aligned} \tag{2.24} $$

The validity of the first four of these relations is demonstrated by Brenner (1964b, 1964c). The last relation is new, its derivation being presented in Appendix A.

All of the uncareted scalars in the above expressions are dimensionless. Though principal interest centers on circumstances where the axially symmetric body possesses fore-aft symmetry, we remark in passing that each of these forms applies (at the center of reaction of the body) even if the body lacks fore-aft symmetry (though the tensors $\hat{\Phi}_{ijk}$ and M_{ijk} are then nonzero). In view of [2.18] the scalar coefficients N and τ are not independent, but rather are connected by the expression

$$ N = \frac{6}{5}\tau. \tag{2.25} $$

It is readily verified that [2.20]–[2.24] satisfy all the general symmetry relations set forth earlier, which must apply irrespective of the geometric symmetry of the body.

As will be demonstrated (cf. the remarks following [4.30]), knowledge of only the five dimensionless scalar coefficients ${}^rK_{\perp}$, N (or τ), Q_1, Q_2 and Q_3 suffices to formulate a completely general theory of the rheological properties of dilute suspensions of identical axisymmetric particles (including the effects of rotary Brownian movement) in arbitrary homogeneous shearing flows. Numerical values of these coefficients may be derived from the solution of the appropriate hydrodynamic problem posed by [2.6]–[2.8] for the body in question. Being dimensionless, these five "fundamental" scalar rheological material constants depend only on the external shape of the particle, but not its size. In Section 3 of this paper, values of these material constants are presented for a variety of differently shaped particles.

* This section was written before the appearance of a paper by Nir & Acrivos (1973), which gives essentially the same relations set forth in [2.20]–[2.24].

In addition to the preceding material tensors, several others derived from them arise in the subsequent rheological theory. These are discussed below.

Define the tensor $B_{ijk} = 2{}^r\hat{K}_{il}^{-1}\hat{\tau}_{ljk}$. Equivalently,

$$B_{ijk} = 2{}^rK_{il}^{-1}\tau_{ljk}. \qquad [2.26]$$

For a body of revolution this reduces to the form

$$B_{ijk} = -(\varepsilon_{ijl}e_l e_k + \varepsilon_{ikl}e_l e_j)B, \qquad [2.27]$$

in which B is the dimensionless scalar,

$$B = 2\tau/{}^rK_\perp. \qquad [2.28a]$$

Alternatively, from [2.25],

$$B = 5N/3{}^rK_\perp. \qquad [2.28b]$$

When B lies in the range $-1 \leq B \leq 1$, we define the "equivalent axis ratio" r_e as*

$$r_e = \left(\frac{1+B}{1-B}\right)^{1/2} \quad (|B| \leq 1). \qquad [2.29]$$

This is equivalent to

$$B = \frac{r_e^2 - 1}{r_e^2 + 1} \quad (|B| \leq 1). \qquad [2.30]$$

On the other hand, when B lies in either of the ranges $\infty > B \geq 1$ or $-\infty < B \leq 1$, we define the symbol r_e as

$$r_e = \left(\frac{B+1}{B-1}\right)^{1/2} \quad (|B| \geq 1). \qquad [2.31]$$

Equivalently,

$$B = \frac{r_e^2 + 1}{r_e^2 - 1} \quad (|B| \geq 1). \qquad [2.32]$$

The question of which of the two ranges, $|B| \leq 1$ or $|B| \geq 1$, B lies in, proves crucial in determining the rotational motion of a neutrally buoyant axisymmetric particle suspended in a simple shearing flow (in the absence of rotary Brownian motion) (Bretherton 1962, Brenner 1972c).

Another derived scalar of importance in the rheological theory is the rotary Brownian diffusion coefficient D_r for rotation of the axisymmetric particle about a transverse axis. This is given by the Stokes–Einstein equation (Brenner 1967) as

$$D_r = kT/\mu_o{}^r\hat{K}_\perp \equiv kT/6V_p\mu_o{}^rK_\perp, \qquad [2.33]$$

where k is Boltzmann's constant and T the absolute temperature. This rotary diffusivity may therefore be calculated from the material constant ${}^rK_\perp$.

* The physical significance of r_e resides in the fact that for a spheroidal particle this parameter is equal to the particle axis ratio r_p of the spheroid. (See [3.14].)

Finally, we define the derived tensor

$$Q^o_{ijkl} = Q_{ijkl} - \tfrac{1}{2}N_{ijm}B_{mkl}. \quad [2.34]$$

For a body of revolution it therefore follows from [2.24], [2.22] and [2.27] that

$$Q^o_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})Q_1 + (\delta_{ij}e_k e_l + \delta_{kl}e_i e_j - 3e_i e_j e_k e_l)Q_2 + (\delta_{jk}e_i e_l + \delta_{il}e_j e_k + \delta_{ik}e_j e_l + \delta_{jl}e_i e_k - 4e_i e_j e_k e_l)Q^o_3, \quad [2.35]$$

with

$$Q^o_3 = Q_3 - \tfrac{1}{2}BN \quad [2.36]$$

a derived material constant. This tensor is of precisely the same form as the $\mathbf{Q}$ tensor in [2.24], with Q^o_3 appearing in place of Q_3.

Though not strictly required in the subsequent rheological theory of axisymmetric bodies, the material constants ${}^t\hat{K}_{\parallel}$ and ${}^t\hat{K}_{\perp}$ arise in closely-related problems dealing with the translational diffusion of anisotropic axisymmetric Brownian particles in homogeneous shearing flows (Brenner & Condiff 1974). In particular, the translational diffusivities of the particle parallel and perpendicular, respectively, to the axis of the body may be expressed in terms of these (dimensional) particle material constants via the Stokes–Einstein relations (Brenner 1967)

$${}^tD_{\parallel} = kT/\mu_o {}^t\hat{K}_{\parallel}, \; {}^tD_{\perp} = kT/\mu_o {}^t\hat{K}_{\perp}. \quad [2.37a, b]$$

Accordingly, it has been deemed worthwhile to tabulate these translational resistance coefficients in Section 3, as well as the material constant ${}^rK_{\parallel}$, which is useful for calculating the rotary diffusivity

$${}^rD_{\parallel} = kT/6V_p\mu_o {}^rK_{\parallel} \quad [2.38]$$

of the particle about its symmetry axis. (The previous rotary coefficient $D_r \equiv {}^rD_{\perp}$ pertains to rotation about an axis perpendicular to this symmetry axis.)

Inequalities satisfied by the material constants

Considerations of the fundamentally positive nature of the energy dissipation arising from the presence of a suspended particle in an otherwise homogeneous shearing flow furnishes lower bounds for certain of the material constants. These are derived in Appendix G, the bounds being as follows:

$${}^t\hat{K}_{\parallel} > 0, \qquad {}^t\hat{K}_{\perp} > 0, \quad [2.39a, b]$$

$${}^rK_{\parallel} > 0, \qquad {}^rK_{\perp} > 0, \quad [2.40a, b]$$

$$Q_1 > 1/5, \quad [2.41]$$

$$Q_1 - Q_2 > 1/5, \quad [2.42]$$

$$Q_1 + Q_3 > 1/5, \quad [2.43]$$

$$Q_1 + Q^o_3 > 1/5. \quad [2.44]$$

Each of these inequalities are satisfied by all of the bodies whose properties are tabulated in Section 3.

3. MATERIAL CONSTANTS FOR VARIOUS AXISYMMETRIC BODIES

In this section values will be given for the fundamental material constants $'\hat{K}_{\parallel}$, $'\hat{K}_{\perp}$, $'K_{\parallel}$, $'K_{\perp}$, N, Q_1, Q_2 and Q_3 for prolate and oblate spheroids, spheres, spherical dumbbells of various aspect ratios, and long slender bodies. From these we will obtain values for the derived material constants Q_3^o, B and r_e, additionally required in the general rheological theory.

Suppose that the suspended particle is neutrally buoyant, i.e. it is force free and couple free. In this case it is found upon setting $F_i = L_i = 0$ in [2.9] and [2.10], and utilizing [2.19] for a centrally-symmetric body, that the translational and angular slip velocities are, respectively,

$$U_i - v_i^o = 0, \tag{3.1}$$

$$\Omega_i - \omega_i = \tfrac{1}{2}B_{ijk}s_{jk}, \tag{3.2}$$

in which [2.26] has been utilized. According to the former relation the center O of the body translates with the velocity of the fluid in its proximity. With use of [2.27] the latter relation specializes for axisymmetric particles to the form

$$\Omega_i - \omega_i = -B\varepsilon_{ijl}e_l e_k s_{jk}, \tag{3.3}$$

wherein B appears as the only material constant.

Substitution of [3.1] and [3.2] into [2.11], and subsequent use of [2.34] yields the following expression for the stresslet:

$$A_{ij} = Q^o_{ijkl}s_{kl}, \tag{3.4}$$

in which Q^o_{ijkl} is given for an axisymmetric body by [2.36].

Spheroids

The five fundamental material constants for an ellipsoid of revolution may be immediately obtained by comparison of [2.21]–[2.24] with equations [3.16], [3.19] and [3.20] of Brenner (1972a) (with $Q_{\rm I}$, $Q_{\rm II}$, $Q_{\rm III}$ of that paper replaced by Q_1, Q_2, Q_3, respectively). Consequently, with

$$r_p = a/b \tag{3.5}$$

the true or particle "axis ratio" of the spheroid (a = polar radius, b = equatorial radius), there is obtained

$$'K_{\perp} = \frac{2(r_p^2+1)}{3(r_p^2\alpha_{\parallel} + \alpha_{\perp})}, \tag{3.6}$$

$$N = \frac{2(r_p^2-1)}{5(r_p^2\alpha_{\parallel} + \alpha_{\perp})}, \tag{3.7}$$

$$Q_1 = \frac{1}{5\alpha'_{\parallel}}, \qquad [3.8]$$

$$Q_2 = \frac{2}{15\alpha'_{\parallel}}\left(1 - \frac{\alpha''_{\parallel}}{\alpha''_{\perp}}\right), \qquad [3.9]$$

$$Q_3 = \frac{1}{5\alpha'_{\parallel}}\left[\frac{r_p(\alpha_{\parallel} + \alpha_{\perp})}{r_p^2\alpha_{\parallel} + \alpha_{\perp}}\left(\frac{\alpha'_{\parallel}}{\alpha'_{\perp}}\right) - 1\right], \qquad [3.10]$$

with

$$\alpha_{\perp} = \frac{r_p^2}{r_p^2 - 1}(1 - \beta), \qquad [3.11a]$$

$$\alpha_{\parallel} = \frac{2}{r_p^2 - 1}(r_p^2\beta - 1), \qquad [3.11b]$$

$$\alpha'_{\perp} = \frac{r_p}{(r_p^2 - 1)^2}(r_p^2 + 2 - 3r_p^2\beta), \qquad [3.11c]$$

$$\alpha'_{\parallel} = \frac{r_p^2}{4(r_p^2 - 1)^2}(3\beta + 2r_p^2 - 5), \qquad [3.11d]$$

$$\alpha''_{\perp} = \frac{r_p^2}{(r_p^2 - 1)^2}[(2r_p^2 + 1)\beta - 3], \qquad [3.11e]$$

$$\alpha''_{\parallel} = \frac{r_p^2}{4(r_p^2 - 1)^2}[2r_p^2 + 1 - (4r_p^2 - 1)\beta], \qquad [3.11f]$$

in which

$$\beta = \frac{\cosh^{-1} r_p}{r_p(r_p^2 - 1)^{1/2}} \quad (r_p > 1), \qquad [3.11g]$$

$$\beta = \frac{\cos^{-1} r_p}{r_p(1 - r_p^2)^{1/2}} \quad (r_p < 1). \qquad [3.11h]$$

The values $r_p > 1$ and $r_p < 1$ refer to prolate and oblate spheroids, respectively.

Use of [2.28b] in conjunction with [3.6] and [3.7] yields

$$B = \frac{r_p^2 - 1}{r_p^2 + 1}. \qquad [3.12]$$

Since the particle axis ratio r_p necessarily lies in the range $0 \leq r_p < \infty$ (being zero for a flat circular disk and approaching infinity for a long needlelike object), it follows that B lies in the range $-1 \leq B \leq 1$, whereupon

$$|B| \leq 1 \qquad [3.13]$$

for a spheroid. Hence, from [2.29] it follows that

$$r_e = r_p, \qquad [3.14]$$

whence the equivalent and true axis ratios coincide for a spheroid.

Finally, we obtain from [2.36] that

$$Q_3^o = \frac{1}{5\alpha_{\parallel}'}\left[\frac{2r_p\alpha_{\parallel}'}{(r_p^2+1)\alpha_{\perp}'} - 1\right], \tag{3.15}$$

wherein the identity $\alpha_{\perp}' = r_p(r_p^2 - 1)^{-1}(\alpha_{\perp} - \alpha_{\parallel})$ has been employed.

The volume of a spheroid is

$$V_p = (4\pi/3)ab^2. \tag{3.16}$$

In addition to these values, for the sake of completeness we note that (Brenner 1967)

$$^rK_{\parallel} = \frac{2}{3\alpha_{\perp}}, \tag{3.17}$$

$$^t\hat{K}_{\parallel} = \frac{16\pi a}{r_p^2(2\beta + \alpha_{\parallel})}, \tag{3.18}$$

$$^t\hat{K}_{\perp} = \frac{16\pi a}{2r_p^2\beta + \alpha_{\perp}}. \tag{3.19}$$

Long thin prolate spheroids. In the limiting case where $r_p \gg 1$, the material constants previously given for the (prolate) spheroid asymptotically approach the following values:

$$^rK_{\perp} = \frac{r_p^2}{3(\ln 2r_p - 0.5)}, \tag{3.20a}$$

$$N = \frac{r_p^2}{5(\ln 2r_p - 0.5)}, \tag{3.20b}$$

$$Q_1 = \frac{2}{5} - \frac{6\ln 2r_p}{5r_p^2}, \tag{3.20c}$$

$$Q_2 = -\frac{r_p^2}{15(\ln 2r_p - 1.5)} + \frac{2}{5}, \tag{3.20d}$$

$$Q_3 = \frac{r_p^2}{10(\ln 2r_p - 0.5)}, \tag{3.20e}$$

and

$$B = 1 - \frac{2}{r_p^2}, \tag{3.20f}$$

$$Q_3^o = \frac{6\ln 2r_p}{5r_p^2}, \tag{3.20g}$$

and

$$^rK_{\parallel} = \frac{2}{3} - \frac{2\ln 2r_p}{3r_p^2}, \tag{3.20h}$$

$$^t\hat{K}_{\parallel} = \frac{4\pi a}{\ln 2r_p - 0.5}, \tag{3.20i}$$

$$'\hat{K}_{\perp} = \frac{8\pi a}{\ln 2r_p + 0.5}. \qquad [3.20j]$$

Spheres. In the special case of a spherical particle (radius $= c$), we have that $a = b = c$, whence

$$r_p = r_e = 1. \qquad [3.21]$$

The α integrals reduce to

$$\alpha_{\parallel} = \alpha_{\perp} = 2/3, \quad \alpha'_{\parallel} = \alpha'_{\perp} = 2/5, \quad \alpha'_{\parallel} = \alpha''_{\perp} = 4/15, \qquad [3.22a, b, c]$$

whereupon the preceding relations adopt the forms

$$'K_{\perp} = 1, \quad N = 0, \quad Q_1 = 1/2, \quad Q_2 = Q_3 = 0, \qquad [3.23]$$

and

$$B = 0, \quad Q_3^o = 0, \qquad [3.24]$$

in addition to

$$'K_{\parallel} = 1, \quad '\hat{K}_{\parallel} = '\hat{K}_{\perp} = 6\pi c. \qquad [3.25]$$

Long slender axisymmetric bodies

Consider an axisymmetric body (possessing fore–aft symmetry) of length $2a$ and cross-sectional radius b at its midpoint. Particular examples of such bodies are prolate spheroids, symmetrical double cones, and circular cylinders of finite length, each of which is depicted in figure 1. Denote by

$$r_p = a/b \qquad [3.26]$$

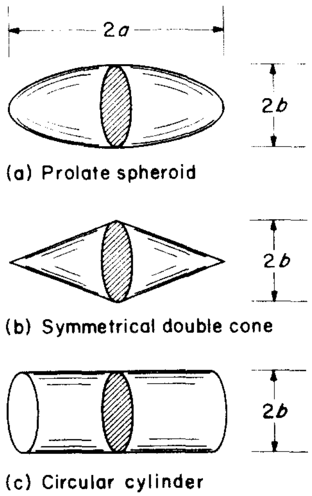

Figure 1. Slender bodies possessing fore–aft symmetry.

the particle axis ratio for the body. A slender body is then defined as one for which

$$r_p \gg 1. \qquad [3.27]$$

Let the parameter t be the distance measured from the midpoint of the body along its symmetry axis, and rendered dimensionless with the length a. Thus, t lies in the range $-1 \leq t \leq 1$, where $t = \pm 1$ corresponds to the endpoints of the particle. Define a dimensionless contour parameter $\sigma \equiv \sigma(t)$ such that $b\sigma$ is the cross-sectional radius the particle at the position t ($t \leq 1$) along its axis. The fore–aft symmetry property of the body is then represented by the fact that $\sigma(t) = \sigma(-t)$. Moreover, by definition, $\sigma(0) = 1$. As examples we have that:

(I) for a spheroid,

$$\sigma = (1 - t^2)^{1/2}; \qquad [3.28]$$

(II) for a symmetrical double cone,

$$\sigma = \begin{cases} 1 - t & \text{for } 1 \geq t \geq 0, \\ 1 + t & \text{for } 0 \geq t \geq -1; \end{cases} \qquad [3.29]$$

(III) for a circular cylinder,

$$\sigma = 1 \quad \text{for all } t. \qquad [3.30]$$

In the results to be cited, particles fall into two general categories—"sharp-ended" or "pointed" bodies and "blunt-ended" bodies. Sharp-ended bodies are those for which:

(i) $\sigma(t)$ is a continuous function of t in the interval $-1 \leq t \leq 1$, [3.31a]

and

(ii) $\sigma(-1) = \sigma(+1) = 0$. [3.31b]

Spheroids and double cones are examples of such bodies, as are spindles too. Blunt-ended bodies are those for which:

(iii) $\sigma(t)$ is piecewise continuous in the interval $-1 \leq t \leq 1$, [3.32a]

and

(iv) $\sigma(t)$ possesses, at most, a finite number of discontinuities in the interval $-1 \leq t \leq 1$. [3.32b]

Circular cylinders are examples of such bodies.

As shown in Appendix B, the following asymptotic formulas, valid for $r_p \gg 1$, apply to *both* classes of bodies:

$$Q_1 = 2/5, \qquad [3.33]$$

$$Q_2 = -\frac{4r_p^2}{45\Lambda(\ln 2r_p + K)}, \qquad [3.34]$$

$$Q_3^o = 0, \qquad [3.35]$$

$$'K_{\parallel} = 2/3, \quad [3.36]$$

$$'\hat{K}_{\parallel} = \frac{4\pi a}{\ln 2r_p + C_o}, \quad [3.37]$$

$$'\hat{K}_{\perp} = \frac{8\pi a}{\ln 2r_p + C_o + 1}. \quad [3.38]$$

The quantities

$$K = -\frac{3}{2} + \frac{3}{4}\int_{-1}^{1} t^2 \ln\left(\frac{1-t^2}{\sigma^2}\right) \mathrm{d}t, \quad [3.39]$$

$$\Lambda = \int_{-1}^{1} \sigma^2 \, \mathrm{d}t, \quad [3.40]$$

and

$$C_o = -\frac{1}{2} + \frac{1}{4}\int_{-1}^{1} \ln\left(\frac{1-t^2}{\sigma^2}\right) \mathrm{d}t \quad [3.41]$$

are numerical constants for a body of given shape. The constant Λ also arises in the expression

$$V_p = \pi a b^2 \Lambda \quad [3.42]$$

for the volume of the slender body.

General formulas for the remaining material constants differ, according as the body is sharp- or blunt-ended. These are tabulated below for each of the two separate cases.

Sharp-ended bodies.

$$'K_{\perp} = \frac{1}{3K_2}\left[\frac{(4/3)r_p^2 + K_2}{\ln r_p} + \frac{(4/3)K_4 r_p^2 + K_3}{(\ln r_p)^2}\right], \quad [3.43]$$

$$N = \frac{1}{5K_2}\left[\frac{(4/3)r_p^2 - K_2}{\ln r_p} + \frac{(4/3)K_4 r_p^2 - K_3}{(\ln r_p)^2}\right], \quad [3.44]$$

$$\frac{r_e}{r_p} = p\left(1 + \frac{q}{\ln r_p}\right), \quad [3.45]$$

$$B = 1 - \frac{2}{p^2 r_p^2}\left(1 - \frac{q}{\ln r_p}\right)^{-2}. \quad [3.46]$$

The quantities

$$K_2 \equiv \Lambda = \int_{-1}^{1} \sigma^2 \, \mathrm{d}t, \quad [3.47]$$

$$K_4 = -\left(\ln 2 - \frac{1}{2}\right) - \frac{3}{4}\int_{-1}^{1} t^2 \ln\left(\frac{1-t^2}{\sigma^2}\right) \mathrm{d}t \equiv -(K + \ln 2 + 1), \quad [3.48]$$

$$p = \left(\frac{4}{3K_2}\right)^{1/2}, \quad q = \frac{1}{2}\left(K_4 - \frac{K_3}{K_2}\right) \tag{3.49a, b}$$

are numerical constants for a body of given shape, as is the constant K_3 too. The latter is given by the expression

$$K_3 = -(1 + \ln 2 + \ln \varepsilon)\int_{-1}^{1} \sigma^2\, dt + \int_{-1}^{1} \sigma^2 \ln \sigma\, dt + \int_{-1}^{1} t\xi(t)\, dt, \tag{3.50a}$$

in which*

$$\xi(t) = \frac{1}{2}\int_{T=-1}^{t-\varepsilon} \frac{1}{t - T}\left[\frac{d}{dT}\sigma^2(T)\right] dT + \frac{1}{2}\int_{T=t+\varepsilon}^{1} \frac{1}{T - t}\left[\frac{d}{dT}\sigma^2(T)\right] dT. \tag{3.50b}$$

Here, $0 < \varepsilon \ll 1$ is an arbitrary (small) positive parameter. The value of K_3 may be shown to be independent of ε.

Blunt-ended bodies.

$$'K_\perp = \frac{4}{9K_2}\left[\frac{r_p^2}{\ln r_p}\left(1 + \frac{K_4}{\ln r_p}\right) + \frac{3L}{8\pi}\right], \tag{3.51}$$

$$N = \frac{4}{15K_2}\left[\frac{r_p^2}{\ln r_p}\left(1 + \frac{K_4}{\ln r_p}\right) - \frac{3L}{8\pi}\right], \tag{3.52}$$

$$\frac{r_e}{r_p} = \left(\frac{8\pi}{3L}\right)^{1/2} \frac{1}{(\ln r_p)^{1/2}}, \tag{3.53}$$

$$B = 1 - \frac{3L \ln r_p}{4\pi r_p^2}, \tag{3.54}$$

in which L is a numerical constant of $O(1)$, which depends critically upon the precise shape of the blunt ends of the body. As yet, this constant has not been calculated theoretically for any bodies. However, from a series of experimental measurements (Anczurowski & Mason 1968) of the equivalent axis ratio r_e as a function of the particle axis ratio r_p for a series of circular cylinders of various aspect ratios (satisfying [3.27]), it has been determined (Cox 1971) via [3.53] that

$$L \approx 5.45 \tag{3.55}$$

for a circular cylinder of finite length.

From the fundamental material constants already tabulated, one can calculate the value of the derived material constant Q_3 by use of [2.36] (as well as τ from [2.25]). Calculated values of the various numerical constants C_i and K_i for the three bodies displayed in figure 1 are tabulated in table 1. By way of confirmation of the general theory, it may be seen that

* For example, for a spheroid, $\sigma^2(T) = 1 - T^2$, we obtain upon integration,

$$\xi(t) = 2t \ln \varepsilon + 2t - t \ln(1 - t^2).$$

Table 1. Numerical constants for various slender bodies.

Body shape / Numerical constant	Sharp-ended bodies: Prolate spheroid	Sharp-ended bodies: Symmetrical double cone	Blunt-ended body: Finite circular cylinder
K_2 or Λ	4/3	2/3	2
$-K$	3/2	$1 - \ln 2$	$(17/6) - \ln 2$
$-C_0$	1/2	$-[\ln 2 - (1/2)]$	$(3/2) - \ln 2$
$-K_3$	$(2/3)(2 \ln 2 - 1)$	$1 - (2/3) \ln 2$	—
$-K_4$	$\ln 2 - (1/2)$	$2 \ln 2$	$1 - \ln 2$
p	1	$2^{1/2}$	—
$-q$	0	$(3/4)(2 \ln 2 - 1)$	—

with use of the coefficients presented in table 1 for the prolate spheroid, all of the asymptotic values tabulated in [3.20] for the long thin prolate spheroid are reproduced by the present theory.*

Dumbbells

As in figure 2, consider a dumbbell composed of equal spheres of radii c, joined by a thin rigid rod of negligible hydrodynamic resistance, with center-to-center spacing $2l$ between spheres. Define the particle axis ratio as

$$r_p = l/c. \qquad [3.56]$$

The case where the spheres touch (tangent-sphere dumbbell) corresponds to the value $r_p = 1$. For the general case where r_p may lie anywhere in the range $1 \leq r_p < \infty$, Wakiya (1971) furnishes an analysis of the motion of such a dumbbell when suspended in a simple shearing

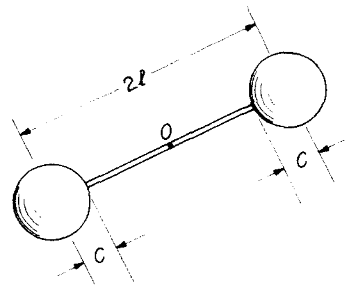

Figure 2. Spherical dumbbell.

* In making the comparison, note that [3.20a] for the spheroid may be expanded into the form

$$'K_\perp = \frac{r_p^2}{3 \ln r_p}\left[1 - \frac{\ln 2 - 0.5}{\ln r_p} + O\left(\frac{1}{\ln r_p}\right)^2\right] \sim \frac{1}{4}\left[\frac{(4/3)r_p^2}{\ln r_p} - \frac{(4/3)(\ln 2 - 0.5)r_p^2}{(\ln r_p)^2}\right],$$

which agrees asymptotically with [3.43] upon utilizing the value $K_4 = -(\ln 2 - 0.5)$ for the spheroid. Similar agreement obtains for the material constant N.

flow. Bipolar and tangent–sphere coordinate systems are employed to solve the appropriate low Reynolds number flow problem (cf. [2.6]–[2.8]) for this special flow.* As discussed in Appendix C, it is possible to extract from Wakiya's analysis the material constants for a dumbbell. This leads to the following values for the five fundamental rheological material constants:

$$'K_\perp = 2(a^2 + b^2), \qquad [3.57]$$

$$N = \tfrac{6}{5}(a^2 - b^2), \qquad [3.58]$$

$$Q_1 = \tfrac{1}{5}e_2, \qquad [3.59]$$

$$Q_2 = \tfrac{1}{5}(e_2 - e_0), \qquad [3.60]$$

$$Q_3^o = \tfrac{1}{5}(e_1 - e_2), \qquad [3.61]$$

as well as the following values for the derived material constants:

$$r_e = a/b, \qquad [3.62]$$

$$B = \frac{a^2 - b^2}{a^2 + b^2}. \qquad [3.63]$$

(cf. [2.30]). Here, a^2, b^2, e_0, e_1 and e_2 are dimensionless numerical constants (Wakiya 1971), tabulated as a function of r_p in table 2.† In turn, these lead to the values of the material

Table 2. Wakiya's dumbbell parameters.

β*	r_p	a^2	b^2	e_0	e_1	e_2
0	1	0.74523	0.18978	4.7760	2.8636	2.3824
0.2	1.020	0.76423	0.19066	4.87	2.886	2.39
0.5	1.1276	0.78073	0.19514	5.405	3.0022	2.428
1.0	1.5431	1.3586	0.20942	7.745	3.3478	2.483
1.5	2.3524	2.6699	0.22521	13.651	3.6798	2.498
2.0	3.7622	6.1142	0.23576	28.399	3.8659	2.500
2.5	6.1323	15.251	0.24175	66.402	3.9478	2.500
3	10.0677	39.718	0.24513	166.559	3.9804	2.500
∞	∞	$(3/8)r_p^2$	1/4	$(3/2)r_p^2$	4	5/2

* The bipolar-coordinate parameter β is defined as $\beta = \cosh^{-1} r_p$.

* Note that Wakiya's analysis exactly takes account of the hydrodynamic interactions among the two spheres comprising the dumbbell. This is in contrast to previous approximate analyses of the dumbbell, where the spheres were assumed to be so far apart (i.e. $r_p \gg 1$) that hydrodynamic interactions among the spheres were regarded as wholly negligible (Bird *et al.* 1971, Brenner & Condiff 1974), or were only taken into account to terms of lowest order in the small parameter r_p^{-1} (Bird & Warner 1971, Stewart & Sørensen 1972).

† Though not listed in the original tabulation of Wakiya (1971), the a^2 and b^2 values were kindly supplied to me by Professor Wakiya, who, at my request, also furnished the more extensive and more accurate results cited in table 2.

The limiting values of these parameters as $r_p \to \infty$ were also furnished to me by Professor Wakiya, who obtained them independently by both a "method of reflections" expansion and an expansion of the exact, bipolar coordinate solution.

Table 3. Rheological material constants for a dumbbell.

β	r_p	${}^rK_\perp$	N	Q_1	Q_2	Q_3^o	Q_3	τ	r_e	B	r_e/r_p
0.0	1	1.87002	0.66654	0.47648	−0.47872	0.09624	0.29422	0.55545	1.9816	0.59406	1.9816
0.2	1.0201	1.90978	0.68828	0.478	−0.496	0.099	0.306	0.57357	2.0021	0.60067	1.9627
0.5	1.1276	1.95174	0.70271	0.4856	−0.5954	0.115	0.326	0.58559	2.0002	0.60007	1.7739
1.0	1.5431	3.1360	1.3790	0.4966	−1.0524	0.173	0.678	1.1492	2.5470	0.73290	1.6506
1.5	2.3524	5.7902	2.9336	0.4996	−2.2306	0.236_4	2.475_0	2.4447	3.4431	0.84442	1.4637
2.0	3.7622	12.6999	7.0541	0.5000	−5.1798	0.27318	3.5383	5.8784	5.0925	0.92574	1.3536
2.5	6.1323	30.986	18.011	0.5000	−12.7804	0.28956	9.0139	15.009	7.9427	0.96878	1.2952
3.0	10.0677	79.926	47.367	0.5000	−32.8118	0.29608	23.689	39.473	12.729	0.98774	1.2643
∞	∞	$(3/4)r_p^2$	$(9/20)r_p^2$	1/2	$-(3/10)r_p^2$	3/10	$(9/40)r_p^2$	$(3/8)r_p^2$	∞	1	$(3/2)^{1/2}$

constants tabulated in table 3.* The values of Q_3 and τ may be obtained from these via [2.36] and [2.25], respectively.

Numerical values of the auxiliary material constants $'\hat{K}_\parallel$ and $'\hat{K}_\perp$ as a function of r_p may be obtained from the bipolar-coordinate calculations of Goldman *et al.* (1966) for the translational motion of two identical spheres in a quiescent fluid, moving parallel and perpendicular, respectively, to their line of centers, when the spheres are each prevented from rotating.† Values of $'K_\parallel$ as a function of r_p may be obtained from the tabulation of Kunesh (1971), derived from the solution of Stokes equations for the equal rotation of two identical spheres about an axis coinciding with their line of centers.‡ A partial tabulation of these results is presented in table 4.

Table 4. Auxiliary material constants for a dumbbell.

β	r_p	$'K_\parallel$ *	$'\hat{K}_\parallel/12\pi c$†	$'\hat{K}_\perp/12\pi c$†
0.0	1.0	0.90154	0.64514	0.72469
0.2	1.0201	0.90562	0.6474	0.7281
0.5	1.1276	0.92504	0.65963	0.74565
1.0	1.5431	0.96752	0.70245	0.79957
1.5	2.3524	0.99050	0.76778	0.86015
2.0	3.7622	0.99766	0.83620	0.90859
2.5	6.1323	0.99946	0.89159	0.94216
3.0	10.0677	0.99988	0.93079	0.96404
∞	∞	1.0	1.0	1.0

* These values were computed from equation [C.21] in Appendix C by Professor Wakiya. They agree with the value tabulated by Kunesh (1971), obtained from the table cited in the last footnote at the bottom of this page.

† See the second footnote at the bottom of this page.

* Independent confirmation of several of these results for the limiting case $r_p = 1$, where the spheres touch, is provided by the work of Majumdar & O'Neill (1972). In our notation these authors give $'K_\perp = 1.8704$ and $\tau = 0.5556$ (i.e. $N = 0.6667$; cf. [2.25]), in close agreement with the values tabulated in table 3. From these values we may also derive $B = 0.59410$ and $r_e = 1.9817$ (cf. [2.28a] and [2.29]), also in close agreement with the corresponding results cited in table 3.

† In the notation of Goldman *et al.* (1966), the expressions for these material constants are given by

$$'K_\parallel = 12\pi c|F_\parallel^*|, \qquad '\hat{K}_\perp = 12\pi c|F_1^{t*}|.$$

Numerical values of $|F_\parallel^*|$ are presented in tables 8 and 8A of these authors. Likewise, $|F_1^{t*}|$ is tabulated in tables 1 and 2 of these authors. The limiting cases where the spheres touch ($r_p = 1$) are treated separately by Majumdar & O'Neill (1972), who obtain values of $'\hat{K}_\parallel/12\pi c = 0.6451$ and $'\hat{K}_\perp/12\pi c = 0.7243$, in good agreement with the limiting results of Goldman *et al.* (1966), cited in table 4 for $r_p = 1$.

‡ By symmetry, this solution is formally identical to that for the symmetrical rotation of a single sphere in proximity to a free surface placed midway between the two spheres, for which numerical calculations are provided by Kunesh (1971), based on the formula (Cox & Brenner 1967)

$$'K_\parallel = \sinh^3\beta \sum_{m=1}^{\infty} (-1)^{m+1} \operatorname{cosech}^3 m\beta,$$

with $\beta = \cosh^{-1} l/c$. The limiting case where the spheres touch is solved separately by Cox & Brenner (1967), Majumdar (1967), and Majumdar & O'Neill (1972).

In utilizing these results it should be noted that the volume of the dumbbell is

$$V_p = 8\pi c^3/3. \qquad [3.64]$$

For the case of touching spheres. Nir & Acrivos (1973; see footnote on page 206) give an independent tabulation of the material constants for the dumbbell. Relationships between the constants appearing in their material tensors and ours are

$$^t\hat{K}_{\parallel} = (a_1 + a_2)/\mu, \quad ^t\hat{K}_{\perp} = a_1/\mu, \quad ^rK_{\parallel} = (b_1 + b_2)/16\pi\mu a_1^3,$$

$$^rK_{\perp} = b_1/16\pi\mu a_1^3, \quad N = 3r_1/40\pi\mu a_1^3, \quad \tau = r_1/16\pi\mu a_1^3, \quad B = 2r_1/b_1,$$

$$Q_1 = c_2/10V_o, \quad Q_2 = -(c_1 + 2c_3)/15V_o, \quad Q_3 = c_3/10V_o,$$

in which μ is the solvent viscosity, a_1 is the radius of each of the identical spheres comprising the dumbbell, and $V_o = 8\pi a_1^3/3$ is the volume of the dumbbell. Use of the tabulated numerical constants $a_1, a_2, b_1, b_2, c_1, c_2, c_3$ and r_1 furnished in their table 2 yields results which agree with ours to at least three significant figures. However, the expressions given for the material tensors R'_{ijk} and R''_{ijk} in their equation [A.2] are in error. In their notation, the correct expressions for these tensors should be

$$R'_{ijk} = -r_1(\varepsilon_{ijl}p_l p_k + \varepsilon_{ikl}p_l p_j),$$

$$R''_{ijk} = r_1(\varepsilon_{ikl}p_l p_j + \varepsilon_{jkl}p_l p_i),$$

where r_1 is the constant defined in their equation [A.3] and tabulated in their table 2. These tensors differ from those reported by Nir & Acrivos (1973) by a minus sign.

Dumbbell composed of "non-interacting" spheres. When $r_p \gg 1$, the two spheres comprising the dumbbell are so far apart that hydrodynamic interactions among them may be neglected in the first approximation. It then becomes possible (see Section 11 of Brenner & Condiff 1974, as well as Appendix D of the present paper) to perform an independent calculation of the material constants, using known results for *isolated* spheres. In this manner we obtain, for the "non-interacting" dumbbell,

$$^rK_{\perp} = (3/4)r_p^2, \qquad [3.65a]$$

$$N = (9/20)r_p^2, \qquad [3.65b]$$

$$Q_1 = O(1), \qquad [3.65c]$$

$$Q_2 = -(3/10)r_p^2, \qquad [3.65d]$$

$$Q_3 = (9/40)r_p^2, \qquad [3.65e]$$

from which may be derived the following values of the secondary constants:

$$B = 1, \quad r_e/r_p = O(1), \quad Q_3^o = O(1). \qquad [3.65f, g, h]$$

These accord with Wakiya's "method of reflection" values, tabulated in table 3. However, the more accurate values of the constants, $Q_1 = 1/2$, $r_e/r_p = (3/2)^{1/2}$ and $Q_3^o = 3/10$, given in table 3, and specified in [3.65] merely by gauge symbols, cannot be calculated by considering the behavior of isolated spheres. Rather, they can only be calculated in numerical value by taking account of first-order interactions between the spheres, as was done by Wakiya via the "method of reflections".

In addition to the above values, we also find for the "non-interacting" dumbbell that

$$ {}^{r}K_{\parallel} = 1, \quad {}^{t}\hat{K}_{\parallel} = 12\pi c, \quad {}^{t}\hat{K}_{\perp} = 12\pi c. \qquad [3.66a, b, c] $$

These derive, respectively, from Kirchhoff's law (Lamb 1932) for the rotation of an isolated sphere about an axis through its center, and from Stokes' law for the translation of an isolated sphere through a quiescent fluid.

Substitution of [3.65c, d, e] into [2.24] yields*

$$ Q_{ijkl} = [\delta_{jk}e_ie_l + \delta_{il}e_je_k + \delta_{ik}e_je_l + \delta_{jl}e_ie_k - (4/3)(\delta_{ij}e_ke_l + \delta_{kl}e_ie_j)](1/2)Q + O(1), \qquad [3.67] $$

in which

$$ Q = (9/20)r_p^2. \qquad [3.68] $$

Note that to terms of $O(r_p^2)$ the fourth order terms, $e_ie_je_ke_l$, in the general Q_{ijkl} tensor in [2.24] have vanished. This makes the "non-interacting" dumbbell suspect as a reasonably general model of an axisymmetric body. Indeed [8.16b] (with $h = 0$), shows that a dilute suspension of such bodies fails to produce a primary normal stress difference. Such atypical behavior is not representative of that exhibited by axisymmetric particles in general.

"First-order" dumbbell. Going beyond the asymptotic values for $r_p \gg 1$ tabulated in table 3, Wakiya (1972) has derived the following, more accurate, asymptotic values of the coefficients a^2 and e_0:

$$ a^2 = \tfrac{3}{8}r_p^2 + \tfrac{9}{64}r_p + \tfrac{155}{512} - \tfrac{399}{4096}r_p^{-1} + O(r_p^{-2}), \qquad [3.69a] $$

$$ e_0 = \tfrac{3}{2}r_p^2 + \tfrac{9}{8}r_p + \tfrac{107}{32} - \tfrac{183}{128}r_p^{-1} + O(r_p^{-2}). \qquad [3.69b] $$

In conjunction with the values (cf. table 3)

$$ b^2 = \tfrac{1}{4} + O(r_p^{-1}), \qquad [3.70a] $$

$$ e_1 = 4 + O(r_p^{-2}), \qquad [3.70b] $$

$$ e_2 = \tfrac{5}{2} + O(r_p^{-4}), \qquad [3.70c] $$

this permits us to obtain analytical expressions for the fundamental material constants to terms of at least $O(1)$ in the aspect ratio r_p.

For purposes of comparing these results with certain related results of Bird *et al.* (1971) in Section 8, we will, however, compute the "first-order" material constants only to $O(r_p)$. Following the notation of Bird *et al.*, define the small dimensionless "interaction" parameter,

$$ h = \frac{3}{8}r_p^{-1} \equiv \frac{3}{8}\frac{c}{l} \ll 1. \qquad [3.71] $$

* The Q_{ijkl} tensor given here is not the same as a comparable tensor given in equation [9.27] of Brenner (1972a). The apparent discrepancy is satisfactorily resolved in Appendix D.

Equations [3.69] may then be written correctly to terms of order r_p as

$$a^2 = \frac{3r_p^2}{8(1-h)} + O(1), \tag{3.72a}$$

$$e_0 = \frac{3r_p^2}{2(1-2h)} + O(1), \tag{3.72b}$$

in which it has been noted that $r_p^2(1+nh) + O(1) = r_p^2(1-nh)^{-1} + O(1)$ $(n = 1, 2)$ to the order of the approximation.

Equations [3.70] and [3.72] lead to the following values of the material constants:

$$'K_\perp = \frac{3r_p^2}{4(1-h)} + O(1), \tag{3.73a}$$

$$N = \frac{9r_p^2}{20(1-h)} + O(1), \tag{3.73b}$$

$$Q_1 = \tfrac{1}{2} + O(h^4), \tag{3.73c}$$

$$Q_2 = -\frac{3r_p^2}{10(1-2h)} + O(1), \tag{3.73d}$$

$$Q_3^0 = \frac{3}{10} + O(h^2), \tag{3.73e}$$

and*

$$\frac{1}{B} = 1 + \frac{4}{3r_p^2}[1 - h + O(h^2)], \tag{3.73f}$$

$$r_e/r_p = (3/2)^{1/2} + O(h), \tag{3.73g}$$

$$Q_3 = \frac{9r_p^2}{40(1-h)} + O(1). \tag{3.73h}$$

The parameter to which the gauge symbols refer is h.

In addition to these first-order interaction values, it also follows that (Brenner 1964a)

$$'K_\parallel = \frac{1}{1 + \frac{64}{27}h^3} + O(h^8), \tag{3.74}$$

as well as (Happel & Brenner 1965)

$$'\hat{K}_\parallel = \frac{12\pi c}{1+2h} + O(h^3), \tag{3.75}$$

* A considerably more accurate value of r_e, derived from [3.62], [3.69a] and [3.70a], is

$$\frac{r_e}{r_p} = \left(\frac{3}{2}\right)^{1/2}\left[1 + \frac{3}{8}r_p^{-1} + \frac{593}{1536}r_p^{-2} + O(r_p^{-3})\right].$$

With use of [2.30] this gives rise to a more accurate expression for B than that tabulated in [3.73f].

and
$$^{t}K_{\perp} = \frac{12\pi c}{1 + h} + O(h^3). \qquad [3.76]$$

Circular disks

The material constants for an infinitesimally thin circular disk of radius b may be obtained from the general results cited in [3.5]–[3.19] for an oblate spheroid, by letting the polar radius a tend to zero. Since the volume V_p of such a disk is zero, results for this case must necessarily be presented in a slightly different form than for prior bodies. In connection with their ultimate use in the basic rheological constitutive equation [4.27], and equations derived therefrom, the appropriate forms for the various material constants are as follows:

$$^{r}\hat{K}_{\perp} = (32/3)b^3, \qquad [3.77a]$$

$$\phi N = -n(16/15)b^3, \qquad [3.77b]$$

$$\phi Q_1 = n(32/45)b^3, \qquad [3.77c]$$

$$\phi Q_2 = n(16/45)b^3, \qquad [3.77d]$$

$$\phi Q_3 = -n(8/45)b^3, \qquad [3.77e]$$

in which n is the number of disks per unit volume. (This number density is related generally to the volume V_p of a particle and to the volume fraction ϕ of suspended particles via the relation $n = \phi/V_p$, in which ϕ and V_p are both zero for circular disks.) Derived material constants are

$$B = -1, \qquad [3.77f]$$

$$r_p = 0, \qquad [3.77g]$$

$$r_e = 0, \qquad [3.77h]$$

$$\hat{\tau} = (16/3)b^3, \qquad [3.77i]$$

$$\phi Q_3^{o} = -n(32/45)b^3. \qquad [3.77j]$$

Auxiliary material constants are

$$^{r}\hat{K}_{\parallel} = (32/3)b^3, \qquad [3.77k]$$

$$^{t}\hat{K}_{\parallel} = 16b, \qquad [3.77l]$$

$$^{t}\hat{K}_{\perp} = (32/3)b. \qquad [3.77m]$$

4. RESUMÉ OF DILUTE SUSPENSION RHEOLOGY THEORY

Consider a dilute, spatially uniform suspension of identical, rigid, force- and couple-free particles (possessing fore–aft symmetry) suspended in a linear homogeneous shear flow. Since the assumption of diluteness implies that the particles are far apart on the average, hydrodynamic interactions among them may be neglected in the first approximation.

The local velocity field in the neighborhood of each particle corresponds to the "undisturbed" flow [2.5]. Due to the collective effect of the disturbance created by all the particles in the suspension, this velocity does not, however, correspond to the mean local velocity of the suspension itself, which quantity will be denoted by $\mathbf{u}$. The mean velocity gradient $\mathbf{G}$ in the suspension is then

$$\mathbf{G} = \nabla \mathbf{u}, \qquad [4.1]$$

which can be decomposed into its symmetric and antisymmetric parts to give

$$\mathbf{S} = \tfrac{1}{2}(\mathbf{G} + \mathbf{G}^{\dagger}) = \mathbf{S}^{\dagger}, \qquad [4.2]$$

and

$$\mathbf{\Lambda} = -\tfrac{1}{2}(\mathbf{G} - \mathbf{G}^{\dagger}) = -\mathbf{\Lambda}^{\dagger}, \qquad [4.3]$$

as the mean rate of strain and mean vorticity dyadics, respectively, in the suspension. In general, these are related to the corresponding undisturbed dyadics, $\mathbf{s}$ and $\boldsymbol{\lambda}$, defined in [2.2] to [2.3], by the relations (Brenner & Condiff 1974)

$$\mathbf{S} = \mathbf{s} + O(\phi), \quad \mathbf{\Lambda} = \boldsymbol{\lambda} + O(\phi), \qquad [4.4a, b]$$

in which ϕ is the volume fraction of suspended particles.

To terms of the first order in ϕ, the mean deviatoric stress $\mathbf{T}$ in the suspension may be calculated from the relation (Brenner & Condiff 1974)

$$\mathbf{T} = 2\mu_o G(\hat{\mathbf{S}} + \tfrac{5}{2}\phi\langle\hat{\mathbf{A}}\rangle^o) + O(\phi^2), \qquad [4.5]$$

with

$$\hat{\mathbf{S}} = \mathbf{S}/G, \quad \hat{\mathbf{\Lambda}} = \mathbf{\Lambda}/G, \qquad [4.6a, b]$$

and $\hat{\mathbf{A}} = \mathbf{A}/G$, in which G is some characteristic shear rate. The mean value denoted by the angular brackets represents an orientational average, defined generally by the expression

$$\langle\psi\rangle^o = \oint \psi(\mathbf{e}) f^o(\mathbf{e})\, \mathrm{d}^2\mathbf{e} \qquad [4.7]$$

for any function $\psi \equiv \psi(\mathbf{e})$ dependent upon the orientation $\mathbf{e}$ of the axisymmetric particle. Particle orientation is represented here by the body-fixed unit vector $\mathbf{e}$ lying along the symmetry axis of the axisymmetric body. Here, $f^o \equiv f^o(\mathbf{e})$ denotes the orientational distribution function, and $\mathrm{d}^2\mathbf{e}$ represents a scalar element of surface area drawn on the unit sphere. Integration is over all orientations. This probability density is normalized to unity:

$$\oint f^o\, \mathrm{d}^2\mathbf{e} = 1. \qquad [4.8]$$

Equation [3.4] for the stresslet $\mathbf{A}$ applies only when rotary Brownian motion is absent. Inclusion of the rotary Brownian movement modifies it to the following form (Brenner & Condiff 1974):

$$A_{ij} = Q^o_{ijkl} s_{kl} - N_{ijk}(\Omega^{\mathrm{Br}}_k + \omega^e_k), \qquad [4.9]$$

provided that the suspended particles are each force free. Here, for axisymmetric bodies (Brenner & Condiff 1974),

$$\mathbf{\Omega}^{Br} = -D_r\mathbf{e} \times \nabla_e f^o \qquad [4.10]$$

is the angular velocity induced by the Brownian motion, with D_r the rotary diffusion coefficient for rotation of the axisymmetric body about a transverse axis, and ∇_e the orientational gradient operator. It will be supposed that the suspended particles are couple free, so that the angular velocity $\boldsymbol{\omega}^e$ induced by the external couples is identically zero.

The orientational distribution function satisfies the conservation relation (Brenner & Condiff 1974)

$$\frac{\partial f^o}{\partial t} + \nabla_e \cdot \mathbf{j}^o = 0, \qquad [4.11]$$

where t is the time. In this expression the rotary flux vector $\mathbf{j}^o \equiv \mathbf{j}^o(\mathbf{e})$ is given by the constitutive relation

$$\mathbf{j}^o = f^o(\mathbf{\Omega} + \mathbf{\Omega}^{Br}) \times \mathbf{e}, \qquad [4.12]$$

with $\mathbf{\Omega}$ the hydrodynamic angular velocity, given by [3.3] for couple-free axisymmetric particles. In the steady state, f^o therefore satisfies the second-order partial differential equation (Brenner & Condiff 1974),

$$\nabla_e^2 f^o = D_r^{-1}\nabla_e \cdot [(\boldsymbol{\lambda} \cdot \mathbf{e} + B\mathbf{s} \cdot \mathbf{e} - B\mathbf{s} \cdot \mathbf{eee})f^o], \qquad [4.13]$$

with $\nabla_e^2 \equiv \nabla_e \cdot \nabla_e$ the angular portion of the Laplace operator on the unit sphere. Consider the analogous function f, in which the undisturbed quantities λ_{ij} and s_{ij} appearing in the above equation are replaced by the mean values Λ_{ij} and S_{ij}, respectively, i.e. f satisfies

$$\nabla_e^2 f = \lambda \nabla_e \cdot [(B^{-1}\hat{\mathbf{\Lambda}} \cdot \mathbf{e} + \hat{\mathbf{S}} \cdot \mathbf{e} - \hat{\mathbf{S}} \cdot \mathbf{eee})f], \qquad [4.14]$$

subject to the normalization condition (cf. [4.8])

$$\oint f \, d^2\mathbf{e} = 1. \qquad [4.15]$$

Here,

$$\lambda = BP, \qquad [4.16]$$

wherein

$$P = G/D_r \qquad [4.17]$$

is the rotary Péclet number; λ therefore represents a weighted Péclet number. In view of [4.4] it follows that

$$f = f^o + O(\phi). \qquad [4.18]$$

Equation [4.10] may now be written as

$$\mathbf{\Omega}^{Br} = \mathbf{\Omega}'^{Br} + O(\phi), \qquad [4.19]$$

wherein

$$\boldsymbol{\Omega}'^{\mathrm{Br}} = -D_r\mathbf{e} \times \nabla_e f. \qquad [4.20]$$

With use of [4.4a] and [4.19], equation [4.9] may now be expressed in the form

$$\hat{A}_{ij} = \hat{A}'_{ij} + O(\phi), \qquad [4.21]$$

wherein

$$\hat{A}'_{ij} = Q^o_{ijkl}\hat{S}_{kl} - G^{-1}N_{ijk}\Omega'^{\mathrm{Br}}_k. \qquad [4.22]$$

Substitution of [4.21] and [4.18] into [4.5] therefore gives

$$\mathbf{T} = 2\mu_o G(\hat{\mathbf{S}} + \tfrac{5}{2}\phi\langle\hat{\mathbf{A}}'\rangle) + O(\phi^2). \qquad [4.23]$$

The angular brackets appearing herein are defined generally as (cf. [4.7]),

$$\langle\psi\rangle = \oint \psi(\mathbf{e})f(\mathbf{e})\,\mathrm{d}^2\mathbf{e}, \qquad [4.24]$$

with f the distribution function satisfying [4.14] and [4.15]. The material tensors Q^o_{ijkl} and N_{ijk} are given generally for axisymmetric particles by [2.35] and [2.22], while the Brownian angular velocity Ω'^{Br}_k is given for such bodies by [4.20]. Use of the expression [4.22] for $\hat{\mathbf{A}}'$ therefore enables us to determine that

$$\langle\hat{\mathbf{A}}'\rangle = 2Q_1\hat{\mathbf{S}} + Q_2\mathbf{I}\hat{\mathbf{S}}:\langle\mathbf{ee}\rangle + 2Q^o_3(\hat{\mathbf{S}}\cdot\langle\mathbf{ee}\rangle + \langle\mathbf{ee}\rangle\cdot\hat{\mathbf{S}})$$
$$-(3Q_2 + 4Q^o_3)\hat{\mathbf{S}}:\langle\mathbf{eeee}\rangle + 2NB\lambda^{-1}(3\langle\mathbf{ee}\rangle - \mathbf{I}), \qquad [4.25]$$

with $\mathbf{I}$ the idemfactor. The term involving the fourth orientational moment may be expressed in terms of second moments via the general theorem (Brenner & Condiff 1974)

$$\hat{\mathbf{S}}:\langle\mathbf{eeee}\rangle = \tfrac{1}{2}[(\hat{\mathbf{S}} + B^{-1}\hat{\mathbf{\Lambda}})\cdot\langle\mathbf{ee}\rangle + \langle\mathbf{ee}\rangle\cdot(\hat{\mathbf{S}} - B^{-1}\hat{\mathbf{\Lambda}})] - \lambda^{-1}(3\langle\mathbf{ee}\rangle - \mathbf{I}), \qquad [4.26]$$

derived from [4.14].

In this manner we may obtain an alternate expression for the mean dimensionless stresslet $\langle\hat{\mathbf{A}}'\rangle$ involving only the second orientational moment $\langle\mathbf{ee}\rangle$. Substitution of the resulting expression into [4.23] then yields, correctly to terms of the first order in ϕ,

$$\frac{\mathbf{T} - 2\mu_o G\hat{\mathbf{S}}}{\phi\mu_o G} = 10Q_1\hat{\mathbf{S}} - \tfrac{15}{2}Q_2(\hat{\mathbf{S}}\cdot\langle\mathbf{ee}\rangle + \langle\mathbf{ee}\rangle\cdot\hat{\mathbf{S}} - \tfrac{2}{3}\mathbf{I}\hat{\mathbf{S}}:\langle\mathbf{ee}\rangle)$$
$$-\tfrac{5}{2}B^{-1}(3Q_2 + 4Q^o_3)(\hat{\mathbf{\Lambda}}\cdot\langle\mathbf{ee}\rangle - \langle\mathbf{ee}\rangle\cdot\hat{\mathbf{\Lambda}})$$
$$+5\lambda^{-1}(3Q_2 + 4Q_3)(3\langle\mathbf{ee}\rangle - \mathbf{I}). \qquad [4.27]$$

The isotropic terms in this expression (i.e. the terms multiplied by $\mathbf{I}$) could be suppressed since they are physically irrelevant for an incompressible suspension. They have been retained, however, at least temporarily, to render the deviatoric stress traceless, i.e.

$$\mathrm{tr}\,\mathbf{T} \equiv \mathbf{I}:\mathbf{T} \equiv T_{kk} = 0, \qquad [4.28]$$

which is the commonly used convention. That this expression is indeed traceless follows in general from [4.1], since both $\hat{\mathbf{S}}$ and $\langle\hat{\mathbf{A}}\rangle$ possess zero trace. (The latter is a consequence of the fact that $A_{ii} = 0$ (Brenner 1972a).)

As follows from [4.24], the second orientational moment is given by

$$\langle \mathbf{ee} \rangle = \oint \mathbf{ee} f(\mathbf{e})\, d^2\mathbf{e}. \qquad [4.29]$$

Subject to the requirement that f be positive, continuous, single-valued, and satisfy the normalization condition [4.15], the solution of [4.14] possesses a unique solution for a prescribed mean velocity gradient $\mathbf{G}$ in the suspension, the solution being of the functional form $f \equiv f(\mathbf{e}; \hat{\mathbf{S}}, \hat{\mathbf{\Lambda}}; B, \lambda)$. In consequence of this, the second orientational moment possesses the general functional form

$$\langle \mathbf{ee} \rangle = \text{function}(\hat{\mathbf{S}}, \hat{\mathbf{\Lambda}}; B, \lambda). \qquad [4.30]$$

Equation [4.27] furnishes the general rheological equation of the suspension of axisymmetric particles to $O(\phi)$. In consequence of [4.30], this equation will generally be highly nonlinear. In general, apart from the parameters μ_o and ϕ, the material constants appearing in this rheological constitutive equation are completely determined by the five fundamental particle constants, ${}^{r}K_{\perp}$, N, Q_1, Q_2 and Q_3 (and the auxiliary constants B, Q_3^o and D_r derivable from these). Since these constants depend only on the shape of the suspended particles, but not their size, the same is true of the deviatoric stress. Values of these constants for a wide variety of axisymmetric bodies are tabulated in Section 3.

It is demonstrated in Appendix G that the time rate of mechanical energy dissipation D per unit volume of the flowing suspension is given by the expression

$$D = \mathbf{T}:\mathbf{S}, \qquad [4.31]$$

and possesses the property that

$$D \geq 0, \qquad [4.32]$$

in which the equality sign holds only when $\mathbf{S} = 0$. Here, $\mathbf{T}$ is the mean deviatoric stress, given by [4.27].

It is sometimes convenient to define a viscosity function η in terms of the energy dissipation by means of the relation

$$\eta = \frac{\mathbf{T}:\mathbf{S}}{2\mathbf{S}:\mathbf{S}} \qquad [4.33]$$

even when the suspension is non-Newtonian. With use of [G.49] this yields the following lower bound on the viscosity of a suspension of rigid axisymmetric particles:

$$\eta > \mu_o(1 + \phi) > 0. \qquad [4.34]$$

This generalizes an earlier result (Brenner 1958) which was only shown to apply in the absence of Brownian motion. Equation [4.34] may be expressed alternatively in terms of the intrinsic viscosity (cf. [5.15]) as

$$[\eta] > 1. \qquad [4.35]$$

In the case where the suspended particles are spherical in shape (cf. [3.23]), [4.27] correctly reduces to Einstein's result,

$$\mathbf{T} = 2\eta\mathbf{S}, \qquad [4.36]$$

with

$$\eta = \mu_o(1 + \tfrac{5}{2}\phi), \qquad [4.37]$$

irrespective of the type of shear flow. The rotary Brownian motion is obviously without effect, as was to be expected.

5. AXISYMMETRIC EXTENSIONAL FLOWS

Perhaps the simplest application of the preceding rheological theory is to the case of uniaxial flows (Trouton 1906), generated—at least in principle—by the extension or compression of a cylindrical thread of fluid. With (x_1, x_2, x_3) a system of Cartesian axis fixed in space, consider the incompressible flow field $\mathbf{u}$ in the suspension,

$$u_1 = -\tfrac{1}{2}Gx_1, \quad u_2 = -\tfrac{1}{2}Gx_2, \quad u_3 = Gx_3, \qquad [5.1]$$

arising from the application of a tensile or compressive force along the x_3 axis. The parameter G represents the fractional rate of elongation of the thread along the x_3 axis. Thus,

$$G > 0 \quad \text{for elongational flows,} \qquad [5.2a]$$

$$G < 0 \quad \text{for contractile flows.} \qquad [5.2b]$$

From [4.1] to [4.3] and [4.6] it follows that

$$\hat{\mathbf{S}} = \tfrac{1}{2}(3\mathbf{i}_3\mathbf{i}_3 - \mathbf{I}), \qquad [5.3]$$

and

$$\hat{\mathbf{\Lambda}} = 0, \qquad [5.4]$$

in which $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ are a right-handed system of unit vectors along (x_1, x_2, x_3). Equation [5.4] is a manifestation of the irrotational nature of the flow.

The unit orientational vector $\mathbf{e}$ may be represented as a unit radial vector in a system of spherical-polar coordinates (r, θ, ϕ):

$$\mathbf{e} = \mathbf{i}_1 \sin\theta\cos\phi + \mathbf{i}_2 \sin\theta \sin\phi + \mathbf{i}_3 \cos\theta. \qquad [5.5]$$

Substitution of [5.3] and [5.4] into [4.14] furnishes the differential equation governing the orientational distribution function. Subject to the normalization condition [4.15], this equation possesses the solution (Brenner & Condiff 1974)

$$f(\theta) = K^{-1} \exp(\tfrac{3}{4}\lambda \cos^2\theta), \qquad [5.6]$$

with $K \equiv K(\lambda)$ the normalization constant

$$K(\xi) = 4\pi\xi^{-1} \exp(\xi^2)D(\xi) \quad \text{for } \lambda > 0, \qquad [5.7a]$$

$$K(\xi) = 4\pi \quad \text{for } \lambda = 0,$$

$$K(\xi) = 2\pi^{3/2}\xi^{-1}\operatorname{erf}\xi \quad \text{for } \lambda < 0, \tag{5.7b}$$

in which
$$D(\xi) = \exp(-\xi^2)\int_0^\xi \exp(z^2)\,dz \tag{5.8}$$

is Dawson's integral (Abramowitz & Stegun 1968; see also [H.41]), and

$$\operatorname{erf}\xi = 2\pi^{-1/2}\int_0^\xi \exp(-z^2)\,dz \tag{5.9}$$

the error function. In these expressions,

$$\xi = |3\lambda/4|^{1/2}. \tag{5.10}$$

The parameter λ may be positive or negative, depending upon the algebraic signs of B and G. For example, in the case of spheroidal particles, [3.5] and [3.12] show that B is positive for prolate spheroids and negative for oblate spheroids. Algebraic signs for G are as indicated in [5.2]. Equations [4.29], [5.5] and [5.6] combine to yield (Brenner & Condiff 1974)

$$\langle \mathbf{ee} \rangle = \tfrac{1}{2}[\mathbf{I} - \mathbf{i}_3\mathbf{i}_3 + (3\mathbf{i}_3\mathbf{i}_3 - \mathbf{I})F(\xi)], \tag{5.11}$$

in which
$$F(\xi) = \frac{1}{2\xi D(\xi)} - \frac{1}{2\xi^2} \quad \text{for } \lambda > 0, \tag{5.12a}$$

$$F(\xi) = \tfrac{1}{3} \quad \text{for } \lambda = 0, \tag{5.12b}$$

$$F(\xi) = \frac{1}{2\xi^2} - \frac{\exp(-\xi^2)}{\pi^{1/2}\xi\operatorname{erf}\xi} \quad \text{for } \lambda < 0. \tag{5.12c}$$

Introduction of [5.3], [5.4] and [5.11] into [4.27] gives

$$\frac{\mathbf{T} - 2\mu_o G\hat{\mathbf{S}}}{\phi\mu_o G} = 5[2Q_1 - \tfrac{3}{2}(F + \tfrac{1}{3})Q_2 + 3\lambda^{-1}(F - \tfrac{1}{3})(3Q_2 + 4Q_3)]\hat{\mathbf{S}} \tag{5.13}$$

for the deviatoric stress. Equivalently,

$$\mathbf{T} = 2\eta\mathbf{S}, \tag{5.14}$$

where, if η is expressed in the terms of the intrinsic viscosity $[\eta]$, defined generally as

$$[\eta] = \lim_{\phi\to 0}\frac{\eta - \mu_o}{\phi\mu_o}, \tag{5.15}$$

then $[\eta]$ is the quantity

$$[\eta] = \tfrac{5}{2}[2Q_1 - \tfrac{3}{2}(F + \tfrac{1}{3})Q_2 + 3\lambda^{-1}(F - \tfrac{1}{3})(3Q_2 + 4Q_3)]. \tag{5.16}$$

Equation [5.14] shows that with respect to axisymmetric extensional or compressive flows, the suspension behaves like a Newtonian fluid possessing a shear-dependent (i e.

G-dependent) viscosity coefficient η. In the case of spheroidal particles [5.14]–[5.16], agree identically with prior results (Brenner 1972a).*

For the limiting case where the Brownian motion is dominant, it follows from either [5.12a] or [5.12c] that $F \sim (1/3)[1 + (\lambda/5) + O(\lambda^2)]$ as $\lambda \to 0\pm$. Hence, in this limit, the "zero-shear" intrinsic viscosity is

$$[\eta]_0 = 5Q_1 - Q_2 + 2Q_3, \qquad [5.17]$$

in which the subscript zero denotes the limiting value as $P \equiv G/D_r \to 0$. From [5.11], in this limit $\langle \mathbf{ee} \rangle \sim \mathbf{I}/3$, corresponding to a random distribution of orientations.

In the opposite case of weak Brownian motion, or equivalently infinite elongational rate ($|G| \to \infty$), $|\lambda| \to \infty$. The algebraic sign of λ depends upon those of B and G. As $\lambda \to +\infty$, $F \sim 1 - (4/3)\lambda^{-1}$, whence

$$[\eta]_{+\infty} = 5(Q_1 - Q_2). \qquad [5.18]$$

On the other hand, $F \sim -(2/3)\lambda^{-1}$ as $\lambda \to -\infty$, whence

$$[\eta]_{-\infty} = 5Q_1 - \tfrac{5}{4}Q_2. \qquad [5.19]$$

As discussed by Brenner (1972a), these two limits, in which rotary diffusion is effectively absent, correspond to preferential alignments of the symmetry axes of the axisymmetric particles relative to the axis of tension or compression of the uniaxial flow [5.1].

Figures 3a and 3b are plots of $[\eta]$ vs the dimensionless "shear" rate G/D_r for spheroidal particles of various aspect ratios r_p. For oblate spheroids ($0 \le r_p < 1$) rheological behavior

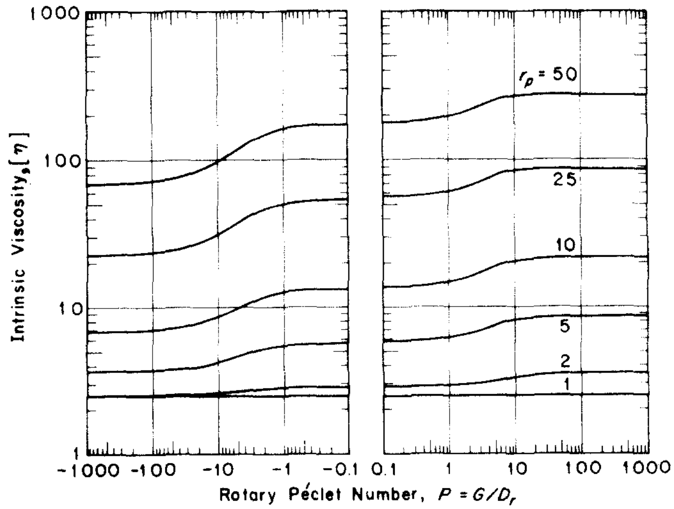

Figure 3a. Variation of intrinsic viscosity with elongation rate for prolate spheroids of various axis ratios suspended in an axisymmetric uniaxial extensional flow.

* In making the comparison, note that λ as defined by Brenner (1972a) is only 3/4 of the value, $\lambda = BG/D_r$, defined in the present paper. Note also that the corresponding expressions for η ($\frac{5}{2}\kappa$ in Brenner 1972a) differ superficially in their appearance, since the theorem [4.26] was not employed to simplify the analysis of Brenner (1972a). However, the two forms can be shown to be identical.

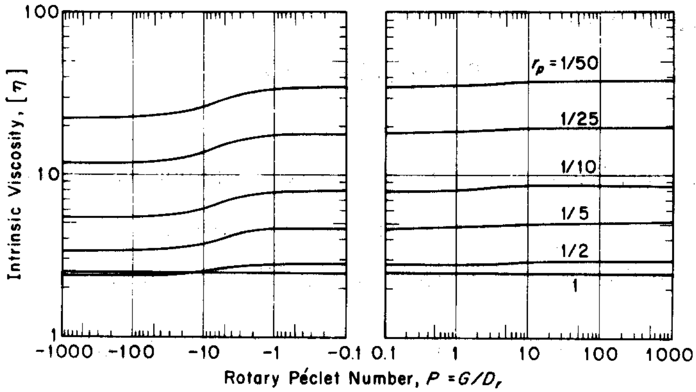

Figure 3b. Variation of intrinsic viscosity with elongation rate for oblate spheroids of various axis ratios suspended in an axisymmetric uniaxial extensional flow.

is of the shear-thickening type, in that $[\eta]$ increases monotonically as G is varied from $-\infty$ to ∞. The same behavior obtains for prolate spheroids whose axis ratios lie in the range $1 < r_p \leq 10.473$ (Clarke 1973). However, for those prolate spheroids characterized by $10.473 < r_p < \infty$, behavior is of the shear-thickening type only up until some value of the dimensional shear rate, whose value depends upon r_p. Beyond this shear rate the behavior is of the shear-thinning type.

For a "non-interacting" dumbbell (cf. [3.65]), [5.16] reduces to

$$[\eta] = \tfrac{9}{8}r_p^2(F + \tfrac{1}{3}), \qquad [5.20]$$

where r_p is the aspect ratio of the dumbbell, defined in [3.56], and $\lambda = G/D_r$ in the present case, since $B = 1$. The agreement of this result with that of Bird *et al.* (1971) for the particular case of extensional flows ($G > 0$) has already been pointed out (Brenner 1972a).

The quantity η defined by [5.14] is identical to the viscosity function defined more generally by [4.33]. Thus, the general inequality, $[\eta] > 1$ (cf. [4.35]), applies in the present circumstances. In particular, with use of [2.41]–[2.43], [5.16] may be demonstrated to satisfy this inequality.

6. PLANE EXTENSIONAL FLOWS

The two-dimensional flow field

$$u_1 = -\tfrac{1}{2}Gx_1, \quad u_2 = \tfrac{1}{2}Gx_2, \quad u_3 = 0, \qquad [6.1]$$

is the two-dimensional (biaxial) counterpart of the axisymmetric (uniaxial) extensional flow [5.1]. This irrotational planar flow can be experimentally realized at the center of a "four roller" apparatus (Taylor 1934, Giesekus 1962b, Chaffey *et al.* 1965). As before, G may be either positive or negative, as in [5.2]. Now, however, the question of the algebraic sign of G is trivial, since G changes sign under the transformation $1 \to 2$ and $2 \to 1$. Hence, in contrast to the results of the prior section, the algebraic sign of G is here devoid of physical significance.

In present circumstances,

$$\mathbf{S} = (1/2)(\mathbf{i}_2\mathbf{i}_2 - \mathbf{i}_1\mathbf{i}_1), \qquad [6.2]$$

and

$$\hat{\Lambda} = 0. \qquad [6.3]$$

The orientational distribution function for this case is (Brenner & Condiff 1974)

$$f(\theta, \phi) = K^{-1} \exp(-\tfrac{1}{4}\lambda \sin^2\theta \cos 2\phi), \qquad [6.4]$$

with angles (θ, ϕ) defined as in [5.5]. Here, $K \equiv K(|\lambda|)$ is the normalization constant

$$K = \int_0^{2\pi}\int_0^{\pi} \exp(-\tfrac{1}{4}\lambda \sin^2\theta \cos 2\phi) \sin\theta \, d\theta \, d\phi, \qquad [6.5]$$

which yields on integration (Brenner & Condiff 1974)

$$K(|\lambda|) = 2^{1/2}\pi^2 I_{1/4}(\tfrac{1}{8}|\lambda|) I_{-1/4}(\tfrac{1}{8}|\lambda|), \qquad [6.6]$$

with I_υ the modified Bessel function of the first kind of order υ.

It is readily shown by symmetry arguments that all "off-diagonal" terms in $\langle \mathbf{ee} \rangle$ are identically zero in the present case, whence $\langle \mathbf{ee} \rangle$ is necessarily of the form

$$\langle \mathbf{ee} \rangle = \mathbf{i}_1\mathbf{i}_1 a_1 + \mathbf{i}_2\mathbf{i}_2 a_2 + \mathbf{i}_3\mathbf{i}_3 a_3, \qquad [6.7]$$

in which

$$a_1 \equiv \langle e_1 e_1 \rangle = \langle \sin^2\theta \cos^2\phi \rangle, \qquad [6.8a]$$

$$a_2 \equiv \langle e_2 e_2 \rangle = \langle \sin^2\theta \sin^2\phi \rangle, \qquad [6.8b]$$

$$a_3 \equiv \langle e_3 e_3 \rangle = \langle \cos^2\theta \rangle, \qquad [6.8c]$$

where, from [5.5],

$$(e_1, e_2, e_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

The coefficient a_3 is not independent of the other two since it is a consequence of [6.8] that

$$a_3 = 1 - (a_1 + a_2). \qquad [6.9]$$

Calculation of the second orientational moment therefore requires evaluation of only a_1 and a_2.

In place of a_1 and a_2 it is convenient to introduce two other parameters, $g(\lambda)$ and $h(\lambda)$, defined by the expressions

$$g = -2(a_1 - a_2) \equiv -2\langle \sin^2\theta \cos 2\phi \rangle, \qquad [6.10]$$

$$h = 2(a_1 + a_2) \equiv 2\langle \sin^2\theta \rangle. \qquad [6.11]$$

In terms of these,

$$a_1 = -\tfrac{1}{4}(g - h), \qquad [6.12a]$$

$$a_2 = \tfrac{1}{4}(g + h), \qquad [6.12b]$$

$$a_3 = 1 - \tfrac{1}{2}h. \qquad [6.12c]$$

From the definition of the bracket integral in [4.24], we find upon utilizing the expression for f in [6.4] that

$$g = -\frac{2}{K}\int_0^{2\pi}\int_0^{\pi} \sin^2\theta\cos 2\phi \exp(-\tfrac{1}{4}\lambda \sin^2\theta\cos 2\phi)\sin\theta\, d\theta\, d\phi, \qquad [6.13]$$

and

$$h = \frac{2}{K}\int_0^{2\pi}\int_0^{\pi} \sin^2\theta \exp(-\tfrac{1}{4}\lambda \sin^2\theta\cos 2\phi)\sin\theta\, d\theta\, d\phi. \qquad [6.14]$$

The first of these integrals may be evaluated by observing from [6.5] that the integral is merely $-4 dK/d\lambda$. Utilization of the expression for K in [6.6], and use of the relation $\lambda = |\lambda| \operatorname{sgn}\lambda$ (where $\operatorname{sgn}\lambda = 1$ for $\lambda > 0$, and $\operatorname{sgn}\lambda = -1$ for $\lambda < 0$), then gives

$$g(\lambda) = (\operatorname{sgn}\lambda)\frac{I_{1/4}(\frac{1}{8}|\lambda|)I'_{-1/4}(\frac{1}{8}|\lambda|) + I_{-1/4}(\frac{1}{8}|\lambda|)I'_{1/4}(\frac{1}{8}|\lambda|)}{I_{1/4}(\frac{1}{8}|\lambda|)I_{-1/4}(\frac{1}{8}|\lambda|)}, \qquad [6.15]$$

in which $I'_\upsilon(x) = dI_\upsilon(x)/dx$. These derivatives may be obtained from the recurrence relation $I'_\upsilon = \frac{1}{2}(I_{\upsilon+1} + I_{\upsilon-1})$. Evaluation of h is discussed in Appendix E, the result ultimately obtained being

$$h(\lambda) = \frac{I_{1/4}(\frac{1}{8}|\lambda|)I_{-1/4}(\frac{1}{8}|\lambda|) + I_{3/4}(\frac{1}{8}|\lambda|)I_{-3/4}(\frac{1}{8}|\lambda|)}{I_{1/4}(\frac{1}{8}|\lambda|)I_{-1/4}(\frac{1}{8}|\lambda|)}. \qquad [6.16]$$

Tabulated values of these modified Bessel functions are available (Luke 1962).

The g and h functions may be expanded for small $|\lambda|$ by means of the modified Bessel function expansion (McLachlan 1955)

$$I_\upsilon(2x) = \frac{x^\upsilon}{\Gamma(\upsilon+1)}\left[1 + \frac{x^2}{\upsilon+1} + \frac{x^4}{2!(\upsilon+1)(\upsilon+2)} + \cdots\right] \quad (|x| \ll 1),$$

in conjunction with the following identities (McLachlan 1955) relating to the factorial function:

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z},$$

$$\Gamma(z) = (z-1)(z-2)\Gamma(z-2),$$

$$z\Gamma(z) = \Gamma(1+z).$$

This eventually gives

$$g(\lambda) = \tfrac{2}{15}\lambda - \tfrac{1}{3150}\lambda^3 + O(\lambda^5) \quad (|\lambda| \ll 1), \qquad [6.17a]$$

and

$$h(\lambda) = \tfrac{4}{3} + \tfrac{1}{315}\lambda^2 + O(\lambda^4) \quad (|\lambda| \ll 1). \qquad [6.17b]$$

Expansions for large $|\lambda|$ may be obtained from the asymptotic expansion of the modified Bessel function (McLachlan 1955)

$$I_\nu(x) = \frac{e^x}{(2\pi x)^{1/2}}\left[1 - \frac{4\nu^2 - 1}{8x} + O\left(\frac{1}{x^2}\right)\right] \quad (|x| \gg 1),$$

thereby obtaining

$$g(\lambda) = 2\,\mathrm{sgn}\,\lambda - \frac{8}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \quad (|\lambda| \gg 1), \qquad [6.18a]$$

$$h(\lambda) = 2 - \frac{4}{|\lambda|} + O\left(\frac{1}{\lambda^2}\right) \quad (|\lambda| \gg 1). \qquad [6.18b]$$

From [6.7] and [6.12] we have in general that

$$\langle \mathbf{ee} \rangle = \tfrac{1}{4}g(\mathbf{i}_2\mathbf{i}_2 - \mathbf{i}_1\mathbf{i}_1) + (\tfrac{3}{4}h - 1)(\mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2) + (1 - \tfrac{1}{2}h)\mathbf{I}. \qquad [6.19]$$

For $|\lambda| \ll 1$, this then yields

$$\langle \mathbf{ee} \rangle = \tfrac{1}{3}[1 - \tfrac{1}{210}\lambda^2 + O(\lambda^4)]\mathbf{I} + \tfrac{1}{30}\lambda[1 - \tfrac{1}{420}\lambda^2 + O(\lambda^4)](\mathbf{i}_2\mathbf{i}_2 - \mathbf{i}_1\mathbf{i}_1) + \tfrac{1}{420}\lambda^2[1 + O(\lambda^2)](\mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2), \qquad [6.20]$$

correct to terms of $O(\lambda^3)$. (With use of [6.2]–[6.3], this agrees identically with the general result for small λ, cited in [7.3].) In the limit where $\lambda \to 0$, this gives $\langle \mathbf{ee} \rangle = \tfrac{1}{3}\mathbf{I}$, corresponding to a random distribution of particle orientations.

For $|\lambda| \gg 1$, [6.18] and [6.19] combine to yield the following asymptotic result:

$$\langle \mathbf{ee} \rangle = \mathbf{i}_1\mathbf{i}_1\left[H(-\lambda) + \{1 - 4H(-\lambda)\}\frac{1}{|\lambda|}\right] + \mathbf{i}_2\mathbf{i}_2\left[H(\lambda) + \{1 - 4H(\lambda)\}\frac{1}{|\lambda|}\right] - \mathbf{i}_3\mathbf{i}_3\frac{2}{|\lambda|} + O\left(\frac{1}{\lambda^2}\right), \qquad [6.21]$$

wherein $H(x)$ is the Heavyside unit function,

$$H(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases} \qquad [6.22]$$

which is equivalent to $H(x) = \tfrac{1}{2}(1 + \mathrm{sgn}\,x)$.

We note from both [6.20] and [6.21], as well as from the more general equation [6.19], that $\langle \mathbf{ee} \rangle$ enjoys the property of being invariant under the simultaneous set of transformations,

$$1 \to 2, \quad 2 \to 1, \quad \lambda \to -\lambda \quad (\text{i.e. } G \to -G). \qquad [6.23]$$

Clearly, such must be the case since $\mathbf{S}$ is invariant under this set of transformations (and $\mathbf{\Lambda} = 0$).

In the limit where $|\lambda| \to \infty$, corresponding to the absence of Brownian motion, [6.21] reduces to

$$\langle \mathbf{ee} \rangle \sim \mathbf{i}_2\mathbf{i}_2 \quad (\lambda \to +\infty), \tag{6.24a}$$

$$\langle \mathbf{ee} \rangle \sim \mathbf{i}_1\mathbf{i}_1 \quad (\lambda \to -\infty). \tag{6.24b}$$

The first of these relations corresponds to the fact that when $BG > 0$, and the particle motion is governed by purely deterministic mechanical principles, the axisymmetric body adopts a unique terminal orientation with its axis of symmetry oriented along the x_2 axis (Brenner 1972c), so that $\mathbf{e} = \mathbf{i}_2$ at steady state. A similar interpretation applies to [6.24b] for the case where $BG < 0$ (Brenner 1972c).

Rheological properties may be calculated from [4.27]. In the present case this yields

$$\mathbf{T} = \mathbf{i}_1\mathbf{i}_1\tau_1 + \mathbf{i}_2\mathbf{i}_2\tau_2, \tag{6.25}$$

upon suppressing the physically irrelevant isotropic term $\mathbf{I}T_{33}$, that would otherwise have appeared. (Thus, $\operatorname{tr}\mathbf{T} \neq 0$ in the present case; cf. [4.28].) The normal stress differences, defined by

$$\tau_1 = T_{11} - T_{33}, \quad \tau_2 = T_{22} - T_{33}, \tag{6.26}$$

are most conveniently expressed in terms of parameters η and σ defined as

$$\tau_1 = G(\sigma - \eta), \quad \tau_2 = G(\sigma + \eta), \tag{6.27}$$

in which

$$\eta = \mu_o + \tfrac{5}{8}\mu_o\phi[8Q_1 - 3h(\lambda)Q_2 + 6\lambda^{-1}g(\lambda)(3Q_2 + 4Q_3)], \tag{6.28a}$$

$$\sigma = -\tfrac{15}{8}\phi\mu_o[g(\lambda)Q_2 + 2\lambda^{-1}\{4 - 3h(\lambda)\}(3Q_2 + 4Q_3)], \tag{6.28b}$$

with values of g and h given generally by [6.15] and [6.16].

The symbol η defined here accords with the general definition of the viscosity function set forth in [4.33]. Accordingly, the viscosity η given by [6.28a] necessarily satisfies the inequality [4.34], as can also be shown directly by use of [2.41]–[2.43].

Rheological results are most conveniently expressed in terms of the intrinsic viscosity

$$[\eta] = (\eta - \mu_o)/\phi\mu_o, \tag{6.29}$$

and the intrinsic "normal stress" function

$$[\sigma] = \sigma/\phi\mu_o. \tag{6.30}$$

For $|\lambda| \ll 1$, [6.17] and [6.28] combine to yield

$$[\eta] = [\eta]_0 - (\lambda^2/210)(2Q_2 + Q_3) + O(\lambda^4), \qquad [6.31]$$

and

$$[\sigma] = (\lambda/7)(Q_3 - Q_2) + O(\lambda^3), \qquad [6.32]$$

in which $[\eta]_0$ is the intrinsic viscosity at zero shear rate, given by [5.17]. These limiting results may also be derived independently as a special case of [7.4].

In the opposite limit, where $|\lambda| \gg 1$, [6.18] combines with [6.28] to give

$$[\eta] = (5/4)(4Q_1 - 3Q_2) + O(|\lambda|^{-1}), \qquad [6.33]$$

and

$$[\sigma] = -(15/4)Q_2 \operatorname{sgn} \lambda + O(|\lambda|^{-1}). \qquad [6.34]$$

Consequently, the intrinsic rheological properties attain limiting asymptotic values as the Péclet number is increased indefinitely. These limiting results apply in the absence of Brownian motion, where purely mechanical considerations (Brenner 1972c) lead to the conclusion that, at steady state, the particles are all aligned parallel to either the x_1 or x_2 axes (cf. [6.24]).

For spherical particles (cf. [3.23] and [3.24]) it follows that $\eta = \mu_o(1 + \frac{5}{2}\phi)$ and $\sigma = 0$, whence $-\tau_1 = \tau_2 = \eta G$. Substitution into [6.25] with use of [6.2] and [4.6a] then correctly yields Einstein's result, [4.36] and [4.37].

Graphs of $[\eta]$ and $[\sigma]$ are presented in figures 4a, 4b, 5a and 5b for prolate and oblate spheroids of various axis ratios r_p as a function of the dimensionless deformation rate $P = G/D_r$. Behavior of the oblate spheroids (figure 4b) is of the "shear-thinning" type, in that the intrinsic viscosity decreases monotonically with increasing deformation rate G. In contrast, for all r_p less than about 15, prolate spheroids (figure 4a) manifest "shear-

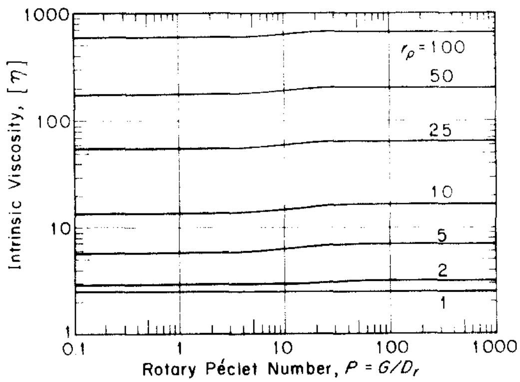

Figure 4a. Variation of intrinsic viscosity $[\eta]$ with elongation rate for prolate spheroids of various axis ratios suspended in a two-dimensional biaxial extensional flow.

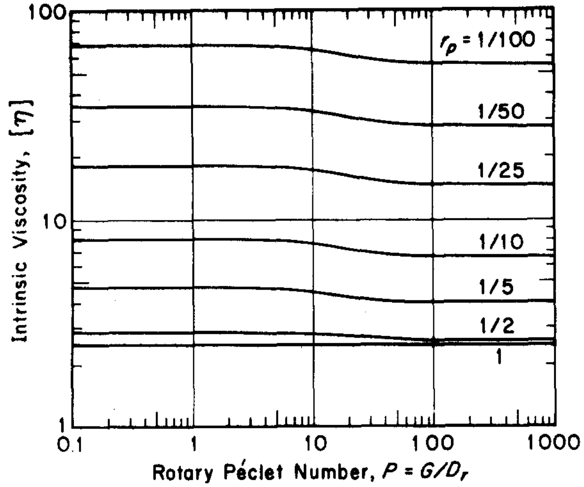

Figure 4b. Variation of intrinsic viscosity $[\eta]$ with elongation rate for oblate spheroids of various axis ratios suspended in a two-dimensional biaxial extensional flow.

thickening" behavior, wherein the intrinsic viscosity increases monotonically with increasing deformation rate. However, for particle axis ratios exceeding about 15, rheological behavior is of the shear-thickening type only at the smaller deformation rates. Beyond some critical shear rate (which depends upon r_p) the behavior reverts to the shear-thinning type. This mixed mode of behavior displayed by prolate spheroids is similar to that pointed out by Clarke (1973) for prolate spheroids in uniaxial extensional flows, as is discussed in Section 5.

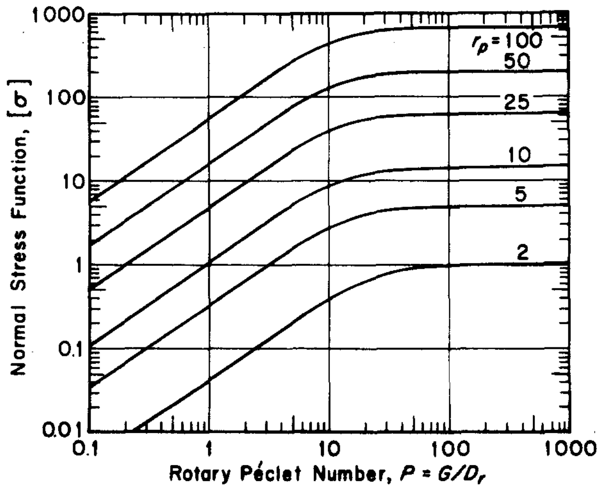

Figure 5a. Variation of normal stress function $[\sigma]$ with elongation rate for prolate spheroids of various axis ratios suspended in a two-dimensional biaxial extensional flow.

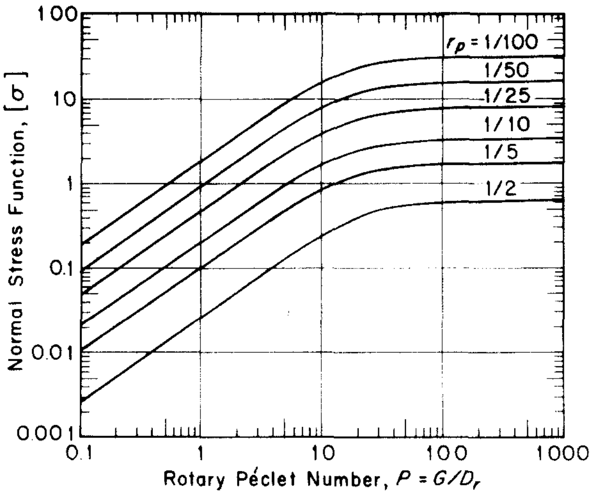

Figure 5b. Variation of normal stress function $[\sigma]$ with elongation rate for oblate spheroids of various axis ratios suspended in a two-dimensional biaxial extensional flow.

7. GENERAL SHEAR FLOWS. SMALL PÉCLET NUMBERS

For values of $|\lambda| \ll 1$ the solution of [4.14] and [4.15] correct to $O(\lambda^3)$ is (Brenner & Condiff 1974)

$$f = (4\pi)^{-1}[1 + \lambda F_1(\mathbf{e}) + \lambda^2 F_2(\mathbf{e}) + \lambda^3 F_3(\mathbf{e})] + O(\lambda^4), \qquad [7.1a]$$

in which

$$F_1(\mathbf{e}) = (1/2)\mathbf{e}\cdot\hat{\mathbf{S}}\cdot\mathbf{e}, \qquad [7.1b]$$

$$F_2(\mathbf{e}) = (1/8)(\mathbf{e}\cdot\hat{\mathbf{S}}\cdot\mathbf{e})^2 - (1/60)\,\mathrm{tr}\,(\hat{\mathbf{S}}^2) - (1/12B)\mathbf{e}\cdot\mathbf{J}_*(\hat{\mathbf{S}})\cdot\mathbf{e}, \qquad [7.1c]$$

$$\begin{aligned} F_3(\mathbf{e}) = {} & (1/48)(\mathbf{e}\cdot\hat{\mathbf{S}}\cdot\mathbf{e})^3 - (1/120)\,\mathrm{tr}\,(\hat{\mathbf{S}}^2)\mathbf{e}\cdot\hat{\mathbf{S}}\cdot\mathbf{e} - (1/630)\,\mathrm{tr}\,(\hat{\mathbf{S}}^3) \\ & - (1/30B)(\mathbf{e}\cdot\hat{\mathbf{S}}\cdot\mathbf{e})\mathbf{e}\cdot\mathbf{J}_*(\hat{\mathbf{S}})\cdot\mathbf{e} - (1/120B)\mathbf{e}\cdot\mathbf{J}_*(\hat{\mathbf{S}}^2)\cdot\mathbf{e} \\ & + (1/72B^2)\mathbf{e}\cdot\mathbf{J}_*^2(\hat{\mathbf{S}})\cdot\mathbf{e}. \end{aligned} \qquad [7.1d]$$

Here, for any dyadic $\mathbf{D}$, $\mathbf{D}^n$ is the dyadic defined as

$$\mathbf{D}^n = \mathbf{D}\cdot\mathbf{D}\cdot\cdots\cdot\mathbf{D} \quad (n \text{ times}). \qquad [7.2a]$$

The dyadic $\mathbf{J}_*$ corresponds to a dimensionless Jaumann derivative,

$$\mathbf{J}_*(\mathbf{D}) = \mathbf{D}\cdot\hat{\mathbf{\Lambda}} - \hat{\mathbf{\Lambda}}\cdot\mathbf{D}, \qquad [7.2b]$$

with $\hat{\mathbf{\Lambda}}$ as defined in [4.6], whereas

$$\mathbf{J}_*^2(\mathbf{D}) = \mathbf{J}_*\{\mathbf{J}_*(\mathbf{D})\} \qquad [7.2c]$$

denotes a multiple Jaumann derivative. The scalar operator

$$\operatorname{tr}\mathbf{D} = \mathbf{I}:\mathbf{D} \qquad [7.2d]$$

represents the trace of the dyadic $\mathbf{D}$.

From [4.29] this yields (Brenner & Condiff 1974)

$$\begin{aligned}\langle \mathbf{ee}\rangle = {} & (1/3)\mathbf{I} + (\lambda/15)\hat{\mathbf{S}} + (\lambda^2/630)[6\hat{\mathbf{S}}^2 - 2\mathbf{I}\operatorname{tr}(\hat{\mathbf{S}}^2) - 7B^{-1}\mathbf{J}_*(\hat{\mathbf{S}})] \\ & + (\lambda^3/56{,}700)[60\hat{\mathbf{S}}^3 - 48\hat{\mathbf{S}}\operatorname{tr}(\hat{\mathbf{S}}^2) - 20\mathbf{I}\operatorname{tr}(\hat{\mathbf{S}}^3) - 135B^{-1}\mathbf{J}_*(\hat{\mathbf{S}}^2) \\ & + 105B^{-2}\mathbf{J}_*^2(\hat{\mathbf{S}})] + O(\lambda^4).\end{aligned} \qquad [7.3]$$

Use of this relation in [4.27] leads to the following expression for the mean deviatoric stress in the suspension:

$$\mathbf{T} = 2\mu_o G\hat{\mathbf{S}} + \phi\mu_o G[\mathbf{T}_0 + \lambda\mathbf{T}_1 + \lambda^2\mathbf{T}_2 + O(\lambda^3)], \qquad [7.4]$$

with

$$\mathbf{T}_0 = 2(5Q_1 - Q_2 + 2Q_3)\hat{\mathbf{S}}, \qquad [7.5a]$$

$$21\mathbf{T}_1 = 12(Q_3 - Q_2)\hat{\mathbf{S}}^2 - 7N\mathbf{J}_*(\hat{\mathbf{S}}), \qquad [7.5b]$$

$$\begin{aligned}630\mathbf{T}_2 = {} & 6Q_2[-10\hat{\mathbf{S}}^3 + \hat{\mathbf{S}}\operatorname{tr}(\hat{\mathbf{S}}^2) + 5B^{-1}\mathbf{J}_*(\hat{\mathbf{S}}^2)] + 5N[7B^{-1}\mathbf{J}_*^2(\hat{\mathbf{S}}) - 6\mathbf{J}_*(\hat{\mathbf{S}}^2)] \\ & + 2Q_3[20\hat{\mathbf{S}}^3 - 16\hat{\mathbf{S}}\operatorname{tr}(\hat{\mathbf{S}}^2) - 15B^{-1}\mathbf{J}_*(\hat{\mathbf{S}}^2)].\end{aligned} \qquad [7.5c]$$

All isotropic terms have been suppressed in these expressions (so that $\operatorname{tr}\mathbf{T} \neq 0$), and [2.36] employed to simplify the final result. Equation [7.4] represents the explicit rheological constitutive equation for a dilute suspension of axisymmetric particles, valid for the case where $|\lambda| \ll 1$. It is, of course, highly nonlinear.

Equation [7.5] can be expressed in a somewhat simpler form by repeated application of the Hamilton–Cayley theorem and extensions thereof (Rivlin 1955), according to which

$$-\mathbf{D}^3 + \mathscr{D}_1\mathbf{D}^2 - \mathscr{D}_2\mathbf{D} + \mathscr{D}_3\mathbf{I} = 0$$

for any dyadic $\mathbf{D}$. Here, $\mathscr{D}_1$, $\mathscr{D}_2$ and $\mathscr{D}_3$ are the scalar invariants of $\mathbf{D}$:

$$\mathscr{D}_1 = \operatorname{tr}\mathbf{D}, \quad \mathscr{D}_2 = \tfrac{1}{2}[\operatorname{tr}\mathbf{D} - \operatorname{tr}(\mathbf{D}^2)], \quad \mathscr{D}_3 = \det\mathbf{D}.$$

This leads, for example, to relations such as

$$\hat{\mathbf{S}}^3 = \mathbf{I}\det\hat{\mathbf{S}} + \tfrac{1}{2}\hat{\mathbf{S}}\operatorname{tr}(\hat{\mathbf{S}}^2),$$

and

$$\hat{\mathbf{\Lambda}}^3 = \tfrac{1}{2}\hat{\mathbf{\Lambda}}\operatorname{tr}(\hat{\mathbf{\Lambda}}^2).$$

Despite the fact that the orientational distribution function [7.1] is known to $O(\lambda^3)$ it is only possible in general to compute the rheological properties to $O(\lambda^2)$, as in [7.4]. This behavior derives from the nature of the last term on the right-hand side of [4.27]. That λ^{-1} appears as a coefficient in this term shows that a calculation of the mean deviatoric stress to $O(\lambda^n)$ presupposes that $\langle \mathbf{ee}\rangle$ and, hence, $f(\mathbf{e})$ be known to $O(\lambda^{n+1})$ $(n = 0, 1, 2, \ldots)$. It is only for bodies characterized by the property $3Q_2 + 4Q_3 = 0$ that this comment is invalid. Of the nonspherical bodies discussed in Section 3, this condition is met only by the "non-interacting" dumbbell (cf. [3.65d] and [3.65e]). For this particular case it is possible to explicitly calculate the term of $O(\lambda^3)$ in [7.4] for the dumbbell. We shall not, however, write down this more exact result here, since it has already been given elsewhere (Brenner & Condiff 1974) to this order.

In the limit where $\lambda = 0$, [7.4] reduces to

$$\mathbf{T} = 2\eta \mathbf{S}, \qquad [7.6]$$

in which

$$\eta = \mu_o[1 + \phi(5Q_1 - Q_2 + 2Q_3)]. \qquad [7.7]$$

Thus, at low shear rates or, more precisely, for $\lambda \to 0$, the suspension displays Newtonian behavior, the viscosity coefficient being given by [7.7]. Alternatively, in terms of the intrinsic viscosity $[\eta]$, defined generally in [5.15], it follows that

$$[\eta]_0 = 5Q_1 - Q_2 + 2Q_3, \qquad [7.8]$$

in which the subscript "zero" refers to the limiting value at zero shear rate.

Equation [7.8] is identical to [5.17], derived for the special case of uniaxial extensional flow, which is irrotational. That the two results are identical is, of course, a consequence of the fact that [7.8] applies to *any* homogeneous shearing flow, irrotational or not.

For the special case of spheroidal particles, possessing values of the material constants tabulated in [3.6]–[3.10], equation [7.8] accords exactly with the analogous "zero shear" (i.e. dominant Brownian motion) result of Simha (1940, 1945), Scheraga (1955). Their computations were, however, performed only for a simple shearing flow, rather than a general homogeneous shearing flow, as in the presentation calculations. Moreover, they employed scalar energy dissipation techniques to compute the intrinsic viscosity, in contrast to the general dynamical tensorial techniques utilized here. For $r_p \gg 1$, the values tabulated in [3.20] for a long thin prolate spheroid make

$$[\eta]_0 = \frac{r_p^2}{15}\left[\frac{3}{\ln 2r_p - 0.5} + \frac{1}{\ln 2r_p - 1.5}\right] + \frac{8}{5} + O\left(\frac{1}{\ln 2r_p}\right), \qquad [7.9]$$

in agreement with the result of Kuhn & Kuhn (1945), derived using scalar energy dissipation methods for the case of a simple shearing field. Analogous results may be derived for any slender body by use of the values of Q_1, Q_2, Q_3 for such bodies, tabulated in Section 3.

For the case of "non-interacting" dumbbells, [7.8] becomes

$$[\eta]_0 = \tfrac{3}{4}r_p^2, \qquad [7.10]$$

(where r_p is given by [3.56]), a well-known result (Brenner 1972a).

8. SIMPLE SHEAR FLOW

Simple shearing flows are of special interest in rheological applications. Consider a suspension subjected to the shear flow

$$\mathbf{u} = \mathbf{i}_2 G x_1, \qquad [8.1]$$

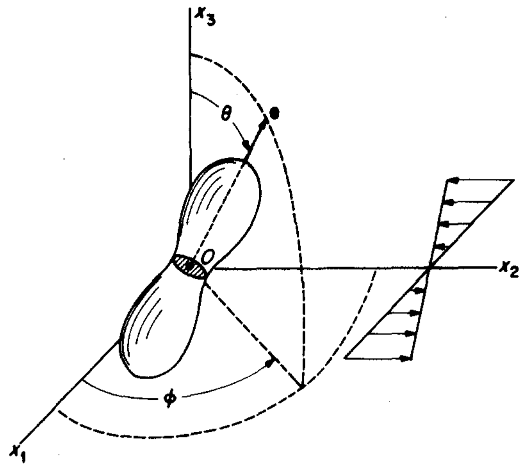

Figure 6. Orientation of a body of revolution in a simple shear flow.

taking place in the $x_1 - x_2$ plane, as in figure 6, G being the shear rate. Dimensionless shear and vorticity dyadics for this flow are

$$\hat{\mathbf{S}} = (1/2)(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1), \quad \hat{\mathbf{\Lambda}} = (1/2)(\mathbf{i}_2\mathbf{i}_1 - \mathbf{i}_1\mathbf{i}_2). \qquad [8.2a, b]$$

Substitution of [5.5] into [4.27] gives rise to the mean deviatoric stress,

$$\mathbf{T} = 2\eta G[\tfrac{1}{2}(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1)] + \mathbf{i}_1\mathbf{i}_1\tau_1 + \mathbf{i}_2\mathbf{i}_2\tau_2. \qquad [8.3]$$

We have suppressed the physically-irrelevant isotropic term $\mathbf{I}T_{33}$ that would otherwise have appeared in this expression. Hence, the deviatoric stress in [8.23] possesses a nonzero trace. In this expression,

$$\eta = T_{12}/G \qquad [8.4]$$

is the viscosity function, whereas τ_1 and τ_2 are the first and second normal stress differences of Coleman *et al.* (1966), defined generally by [6.26]. In [8.3],

$$[\eta] = 5Q_1 - \tfrac{15}{4}Q_2\langle\sin^2\theta\rangle - \tfrac{5}{4}B^{-1}(3Q_2 + 4Q_3^\circ)\langle\sin^2\theta\cos 2\phi\rangle + \tfrac{15}{2}\lambda^{-1}(3Q_2 + 4Q_3)\langle\sin^2\theta\sin 2\phi\rangle, \qquad [8.5]$$

$$[\tau_1] = 5[\tfrac{3}{4}(B^{-1} - 1)Q_2 + B^{-1}Q_3^\circ]\langle\sin^2\theta\sin 2\phi\rangle - 15\lambda^{-1}(3Q_2 + 4Q_3)(1 - \tfrac{3}{2}\langle\sin^2\theta\rangle - \tfrac{1}{2}\langle\sin^2\theta\cos 2\phi\rangle), \qquad [8.6]$$

$$[\tau_2] = -5[\tfrac{3}{4}(B^{-1} + 1)Q_2 + B^{-1}Q_3^\circ]\langle\sin^2\theta\sin 2\phi\rangle - 15\lambda^{-1}(3Q_2 + 4Q_3)(1 - \tfrac{3}{2}\langle\sin^2\theta\rangle + \tfrac{1}{2}\langle\sin^2\theta\cos 2\phi\rangle), \qquad [8.7]$$

wherein $[\eta]$ is the intrinsic viscosity, defined in [5.15], and

$$[\tau_1] = \lim_{\phi\to 0}\frac{\tau_1}{\phi\mu_o G}, \quad [\tau_2] = \lim_{\phi\to 0}\frac{\tau_2}{\phi\mu_o G}, \qquad [8.8a, b]$$

are the "intrinsic normal stress differences". In these expressions,

$$\langle \sin^2\theta\rangle = \int_0^{2\pi}\int_0^{\pi} \sin^2\theta\, f(\theta,\phi)\sin\theta\, d\theta\, d\phi, \text{ etc.} \qquad [8.9]$$

Relations similar to [8.5]–[8.7] have been given by Hinch & Leal (1972) for the special case of spheroidal particles.* However, their expressions involve the additional moments, $\langle \sin^4\theta \sin 2\phi\rangle$, $\langle \sin^4\theta \sin^2 2\phi\rangle$ and $\langle \sin^4\theta \sin 4\phi\rangle$. These may, however, be expressed in terms of the three lower-order moments appearing in [8.5]–[8.7] by means of the following identities, derived from the general theorem [4.26] applied to the shear flow [8.2]:

$$\langle \sin^4\theta \sin 2\phi\rangle = \langle \sin^2\theta \sin 2\phi\rangle - 2\lambda^{-1}(3\langle \sin^2\theta\rangle - 2), \qquad [8.10a]$$

$$\langle \sin^4\theta \sin^2 2\phi\rangle = \langle \sin^2\theta\rangle + B^{-1}\langle \sin^2\theta \cos 2\phi\rangle - 6\lambda^{-1}\langle \sin^2\theta \sin 2\phi\rangle, \qquad [8.10b]$$

$$\langle \sin^4\theta \sin 4\phi\rangle = -2B^{-1}\langle \sin^2\theta \sin 2\phi\rangle - 12\lambda^{-1}\langle \sin^2\theta \cos 2\phi\rangle. \qquad [8.10c]$$

The quantity η defined by [8.3] accords with the general definition of the viscosity function in [4.33]. Accordingly, [4.35] shows that the intrinsic viscosity $[\eta]$, given in present circumstances by [8.5], can never be less than unity. Inequalities [2.41]–[2.44] may be invoked to provide an independent demonstration of the fact that $[\eta] > 1$, irrespective of the shapes of the suspended particles.

Dominant Brownian motion

For the case where $|\lambda| \ll 1$, it follows from [7.3] and [8.2] that

$$\langle \mathbf{ee}\rangle = \tfrac{1}{3}\mathbf{I} + \tfrac{1}{30}\lambda(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1) + \tfrac{1}{630}\lambda^2[\tfrac{1}{2}(1 - 7B^{-1})\mathbf{i}_1\mathbf{i}_1 + \tfrac{1}{2}(1 + 7B^{-1})\mathbf{i}_2\mathbf{i}_2 - \mathbf{i}_3\mathbf{i}_3]$$
$$- \tfrac{1}{37{,}800}\lambda^3(3 + 35B^{-2})(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1) + O(\lambda^4), \qquad [8.11]$$

in which $\mathbf{I} = \mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2 + \mathbf{i}_3\mathbf{i}_3$. Use of [5.5] in conjunction with some elementary trigonometric identities then gives for this case,

$$\langle \sin^2\theta\rangle = \tfrac{2}{3} + \tfrac{1}{630}\lambda^2 + O(\lambda^4), \qquad [8.12a]$$

$$\langle \sin^2\theta \sin 2\phi\rangle = \tfrac{1}{15}[\lambda - \tfrac{1}{1260}(3 + 35B^{-2})\lambda^3 + O(\lambda^5)], \qquad [8.12b]$$

$$\langle \sin^2\theta \cos 2\phi\rangle = -\tfrac{1}{90}B^{-1}\lambda^2 + O(\lambda^4). \qquad [8.12c]$$

Substitution of these into [8.5]–[8.7] yields

$$[\eta] = [\eta]_0 - \tfrac{1}{1260}(12Q_2 + 6Q_3 + 35B^{-1}N)\lambda^2 + O(\lambda^4), \qquad [8.13a]$$

$$[\tau_1] = [\tfrac{1}{7}(Q_3 - Q_2) - \tfrac{1}{6}N]\lambda + O(\lambda^3), \qquad [8.13b]$$

$$[\tau_2] = [\tfrac{1}{7}(Q_3 - Q_2) + \tfrac{1}{6}N]\lambda + O(\lambda^3), \qquad [8.13c]$$

with $[\eta]_0$ given by [7.8].

For the spheroid these reduce to previously known results (Brenner & Condiff 1974).

* If we identify their coordinates (x, y, z) as our (x_1, x_2, x_3), then their angles (θ_1, ϕ_1) are identical to our angles (θ, ϕ). However, their (τ_1, τ_2) are our (τ_2, τ_1), defined in [6.26].

For the "non-interacting" dumbbell they adopt the forms*

$$[\eta] = \tfrac{3}{4}r_p^2[1 - \tfrac{1}{70}P^2 + O(P^4)], \qquad [8.14a]$$

$$[\tau_1] = 0, \qquad [8.14b]$$

$$[\tau_2] = \tfrac{3}{20}r_p^2 P + O(P^3), \qquad [8.14c]$$

where $P = G/D_r$ is the rotary Péclet number. These results are, in fact, well known (cf. Brenner & Condiff 1974, and earlier references cited therein).

"First-order" dumbbell in simple shear flow

Substitution into [8.5]–[8.7] of the material Q constants given by [3.73] yields the general expressions

$$[\eta] = \frac{3r_p^2}{4(1-h)}\left\{\frac{\eta - \eta_s}{n_o kT\lambda_h}\right\} + O(1), \qquad [8.15a]$$

$$[\tau_1] = \frac{r_p^2 P}{8(1-h)}\left\{\frac{\beta}{n_o kT\lambda_h^2}\right\} + O(1), \qquad [8.15b]$$

$$[\tau_2] - [\tau_1] = \frac{r_p^2 P}{8(1-h)}\left\{\frac{\Theta}{n_o kT\lambda_h^2}\right\} + O(1), \qquad [8.15c]$$

with the terms in curly brackets given by

$$\frac{\eta - \eta_s}{n_o kT\lambda_h} = \frac{3}{2}\left(\frac{1-h}{1-2h}\right)(\langle\sin^2\theta\rangle + \langle\sin^2\theta\cos 2\phi\rangle) - \frac{9}{P}\left(\frac{h}{1-2h}\right)\langle\sin^2\theta\sin 2\phi\rangle, \qquad [8.16a]$$

$$\frac{\beta}{n_o kT\lambda_h^2} = \frac{108}{P^2}\left(\frac{h}{1-2h}\right)(1 - \tfrac{3}{2}\langle\sin^2\theta\rangle - \tfrac{1}{2}\langle\sin^2\theta\cos 2\phi\rangle), \qquad [8.16b]$$

$$\frac{\Theta}{n_o kT\lambda_h^2} = \frac{18}{P}\left(\frac{1-h}{1-2h}\right)\langle\sin^2\theta\sin 2\phi\rangle + \frac{108}{P^2}\left(\frac{h}{1-2h}\right)\langle\sin^2\theta\cos 2\phi\rangle. \qquad [8.16c]$$

These expressions are valid for the case where $r_p \gg 1$, i.e. where the small interaction parameter h defined in [3.71] satisfies the condition $h \ll 1$. The parameter to which the gauge symbol O in [8.15] refers is h. In these equations, P is the rotary Péclet number, $P = G/D_r$, in which (cf. [2.33] and [3.73a]),

$$D_r = \frac{kT(1-h)}{12\pi\mu_o cl^2}. \qquad [8.17]$$

* The normal stress difference $[\tau_1]$ for the non-interacting dumbbell is identically zero to all orders in P. Since the deviatoric stress for non-interacting dumbbells can be calculated to $O(\lambda^3)$ (cf. the comments in the paragraph following [7.5]), it is possible to write down a somewhat more accurate expression for $[\tau_2]$ than is given in [8.14c], namely

$$[\tau_2] = \frac{3}{20}r_p^2\left[P - \frac{19}{630}P^3 + O(P^5)\right].$$

Equations [8.15]–[8.16] agree identically with the results of Stewart & Sørensen (1972), when converted to equivalent notation.* That two such vastly different methods of derivation should yield precisely the same results is remarkable.

For the case where $P \ll 1$, we find upon setting $B = 1$ and $\lambda = P$ in [8.12], and introducing those results into [8.16], that

$$\frac{\eta - \eta_s}{n_o kT\lambda_h} = \left(\frac{1-h}{1-2h}\right)\left[1 - \frac{1}{70}P^2 + O(P^4)\right] - \frac{3}{5}\left(\frac{h}{1-2h}\right)\left[1 - \frac{19}{630}P^2 + O(P^4)\right], \qquad [8.18a]$$

$$\frac{\beta}{n_o kT\lambda_h^2} = \frac{12}{35}\left(\frac{h}{1-2h}\right)[1 + O(P^2)], \qquad [8.18b]$$

$$\frac{\Theta}{n_o kT\lambda_h^2} = \frac{6}{5}\left(\frac{1-h}{1-2h}\right)\left[1 - \frac{19}{630}P^2 + O(P^4)\right] - \frac{6}{5}\frac{h}{1-2h}[1 + O(P^2)], \qquad [8.18c]$$

in exact agreement with Bird & Warner (1971), upon replacing their $\lambda_h\kappa$ by $P/6$.

Slender bodies and dumbbells characterized by $B = 1$. Arbitrary Péclet numbers

The partial differential equation governing the orientational distribution of particles immersed in the simple shear flow [8.2] is, from [4.14] and [5.5], governed by the relations

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = P\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(f\dot{\theta}\sin\theta) + \frac{\partial}{\partial\phi}(f\dot{\phi})\right], \qquad [8.19]$$

subject to the normalization condition

$$\int_0^{2\pi}\int_0^{\pi} f\sin\theta\,d\theta\,d\phi, \qquad [8.20]$$

with $f \equiv f(\theta, \phi; B, P)$, in which $P = G/D_r$, and

$$\dot{\theta} = \tfrac{1}{4}B\sin 2\theta\sin 2\phi, \qquad [8.21a]$$

$$\dot{\phi} = \tfrac{1}{2}(1 + B\cos 2\phi), \qquad [8.21b]$$

are the appropriate dimensionless components of the hydrodynamic angular velocity of the particle.

* In effecting the comparison one must utilize the following notational equivalences:

Stewart & Sørensen (1972), and Bird *et al.* (1971)	θ	ϕ	(x, y, z)	τ_{yx}	$\tau_{xx} - \tau_{yy}$	$\tau_{yy} - \tau_{zz}$	κ
This paper	θ	$(\pi/2) - \phi$	$(x_2, x_1, -x_3)$	$-T_{12}$	$-(T_{22} - T_{11})$	$-(T_{11} - T_{33})$	G

Stewart & Sørensen (1972), and Bird *et al.* (1971)	L	r	η_s	n_o	$\beta\kappa^2$	$\Theta\kappa^2$	λ_h
This paper	$2l$	c	μ_o	$3\phi/8\pi c^3$	$\tau_1 \equiv T_{11} - T_{33}$	$\tau_2 - \tau_1 \equiv T_{22} - T_{11}$	$1/6D_r$

with D_r given by [8.17]. Moreover, in our notation, $J = 1$ in the Stewart & Sørensen (1972) and Bird *et al.* (1971) references.

Equations [3.46], [3.54], [3.65] and [3.73] reveal that, for all the "slender" bodies (including spherical dumbbells with first-order interactions) whose properties are tabulated in Section 3,

$$B \to 1 \quad \text{for } r_p \gg 1. \qquad [8.22]$$

Upon setting $B = 1$ in [8.21], equation [8.19] adopts the form

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = P\left[\frac{\sin\phi\cos\phi}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\cos\theta f) + \frac{\partial}{\partial\phi}(\cos^2\phi f)\right]. \qquad [8.23]$$

where, now, $f \equiv f(\theta, \phi; P)$. This equation is identical to the steady-state equation of Bird & Warner (1971); (cf. the notational equivalences in the footnote on page 244), numerical solutions of which are provided over the complete Péclet number range, $0 \leq P < \infty$, by Stewart & Sørensen (1972). The latter's results may therefore be employed to obtain values for the three goniometrical factors, $\langle\sin^2\theta\rangle$, $\langle\sin^2\theta\sin 2\phi\rangle$ and $\langle\sin^2\theta\cos 2\phi\rangle$, as a function of P, for the special case where $B = 1$. In conjunction with [8.5]–[8.7], such information may be employed to calculate the complete rheological properties of dilute suspensions of various slender bodies (i.e. bodies for which $r_p \gg 1$) suspended in simple shearing flows.

Stewart & Sørensen do not directly tabulate the goniometric factors. Rather, they tabulate numerical values of the three viscometric functions in [8.16] at various values of P for the special cases where $h = 0$ and $h = 3/8$. When $h = 3/8$, simultaneous solution of [8.16a] to [8.16c] for the goniometrical factors, yields

$$\langle\sin^2\theta\rangle = [162(25P^2 + 486)]^{-1}[5P^2(810 - 108X - 5P^2Y) + 162(324 - 2P^2Y - P^2Z)],$$

$$\langle\sin^2\theta\cos 2\phi\rangle = 2 - \frac{P^2Y}{81} - 3\langle\sin^2\theta\rangle,$$

$$\langle\sin^2\theta\sin 2\phi\rangle = \frac{PZ}{45} - \frac{18}{5P}\langle\sin^2\theta\cos 2\phi\rangle,$$

in which X, Y and Z, respectively, represent the left-hand sides of [8.16a, b, c] at $h = 3/8$. Use of the tabulated values of X, Y and Z presented by Stewart & Sørensen (1972), in conjunction with the above three equations, then leads to the values of the goniometrical factors noted in table 5. These were confirmed by computing X, Y and Z at $h = 0$ from [8.16] using the values in table 5, and comparing the results so obtained with those tabulated by Stewart & Sørensen at $h = 0$.

In the particular case of the "first-order" dumbbell, the large Péclet number values tabulated in table 5 are limited in their range of applicability to circumstances in which the dimensionless interaction parameter h is sufficiently small to satisfy the inequality (cf. [9.22])

$$h^{-1} \gg P^{1/3} \gg 1.$$

Thus, Stewart & Sørensen's (1972) use of the value of $h = 3/8$ (corresponding to the case where the spheres touch) may lead to significant errors in the theoretical predictions at the

Table 5. Goniometric factors for bodies characterized by $B = 1$.*

P	$\langle \sin^2 \theta \rangle$	$\langle \sin^2 \theta \cos 2\phi \rangle$	$\langle \sin^2 \theta \sin 2\phi \rangle$
0	0.66667	0.00000	0.00000
0.6	0.66723	−0.00395	0.03957
0.75	0.66754	−0.00612	0.04916
1.5	0.67000	−0.02311	0.09376
2.0	0.67231	−0.03886	0.11943
3.0	0.67788	−0.07592	0.15981
4.5	0.68700	−0.13328	0.19684
6.0	0.69569	−0.18394	0.21557
9.0	0.71055	−0.26154	0.22825
12.5	0.72436	−0.32450	0.22840
15	0.73237	−0.35823	0.22548
20	0.74568	−0.40938	0.21782
25	0.75619	−0.44680	0.21001
30	0.76465	−0.47573	0.20276
40	0.77816	−0.51879	0.19042
50	0.7886	−0.5499	0.1804
60	0.7967	−0.5738	0.1722
90	0.8151	−0.6232	0.1541
120	0.8274	−0.6547	0.1418
180	0.8435	−0.6944	0.1255
300	0.866	−0.741	0.107
420	0.876	−0.765	0.096
600	0.889	−0.791	0.086
$P \to \infty$†	$1 - 0.974P^{-1/3}$	$-1 + 1.796P^{-1/3}$	$0.727P^{-1/3}$

* The corresponding goniometric factors for the case $B = -1$ may be obtained directly from the values tabulated herein by means of the transformations noted in equations [8.26].

† See Appendix F.

larger Péclet numbers, even apart from the fact that a first-order hydrodynamic interaction theory would be unlikely to yield correct rheological results for small separation distances.

It should be emphasized that the results cited in table 5 apply to any axisymmetric body for which $B = 1$, and is not limited to dumbbells composed of distant spheres. When B is not exactly unity, but is near to it, the analysis of Section 9 (cf. [9.12] and [9.19]) shows that the larger Péclet number values tabulated in table 5 are valid only when

$$(1 - |B|)^{-1/2} \gg P^{1/3} \gg 1, \qquad [8.24]$$

in which the goniometric factors are those tabulated in table 5.

Circular disks and other bodies for which $B = -1$. Arbitrary Péclet numbers

As can be verified by direct substitution of [8.19]–[8.21] governing the orientational distribution function remain invariant under the set of simultaneous transformations,

$$B \to -B, \qquad [8.25a]$$

$$\theta \to \theta, \qquad [8.25b]$$

$$\phi \rightarrow (\pi/2) + \phi. \tag{8.25c}$$

Hence, Stewart & Sørensen's numerical solution for the case $B = 1$ may be utilized for the case where $B = -1$ by replacing the goniometric factors in table 5 as follows:

$$\langle \sin^2 \theta \rangle \rightarrow \langle \sin^2 \theta \rangle, \tag{8.26a}$$

$$\langle \sin^2 \theta \cos 2\phi \rangle \rightarrow -\langle \sin^2 \theta \cos 2\phi \rangle, \tag{8.26b}$$

$$\langle \sin^2 \theta \sin 2\phi \rangle \rightarrow -\langle \sin^2 \theta \sin 2\phi \rangle. \tag{8.26c}$$

From [2.29] it is seen that $B = -1$ is equivalent to

$$r_e = 0, \tag{8.27}$$

which is the value appropriate to an infinitesimally thin circular disk (cf. [3.14] and [3.15]). Substitution of the material constants cited in [3.77] into [8.5]–[8.7] leads to the following expressions for the three viscometric functions appropriate to a dilute suspension of circular disks of radii b suspended in a simple shear flow:

$$\frac{\eta - \eta_o}{\mu_o n b^3} = \frac{8}{9}\left[4 - \frac{3}{2}\langle \sin^2 \theta \rangle - \frac{5}{2}\langle \sin^2 \theta \cos 2\phi \rangle - \frac{3}{P}\langle \sin^2 \theta \sin 2\phi \rangle\right], \tag{8.28a}$$

$$\frac{\tau_1}{\mu_o G n b^3} = \frac{8}{9}\left[\langle \sin^2 \theta \sin 2\phi \rangle + \frac{6}{P}\left(1 - \frac{3}{2}\langle \sin^2 \theta \rangle - \frac{1}{2}\langle \sin^2 \theta \cos 2\phi \rangle\right)\right], \tag{8.28b}$$

$$\frac{\tau_2}{\mu_o G n b^3} = \frac{16}{9}\left[-2\langle \sin^2 \theta \sin 2\phi \rangle + \frac{3}{P}\left(1 - \frac{3}{2}\langle \sin^2 \theta \rangle + \frac{1}{2}\langle \sin^2 \theta \cos 2\phi \rangle\right)\right], \tag{8.28c}$$

in which the goniometrical factors are those derivable from table 5 via [8.26]. Here, n is the number of disks per unit volume of suspension. The relation $\lambda = -P$ for circular disks has been employed in the derivation, in which the rotary diffusion coefficient is

$$D_r = 3kT/32\mu_o b^3. \tag{8.29}$$

Goniometric factors for the general case

Scheraga *et al.* (1951, 1955) numerically solved equations [8.19]–[8.21] governing the orientational distribution function for axisymmetric bodies immersed in a simple shear flow, for values of $0 \leq P \leq 200$ and for B lying in the range $0 \leq B \leq 1$ (i.e. $1 \leq r_e < \infty$). Though Scheraga *et al.* (1951, 1955) had in mind only spheroidal particles, characterized by axis ratios r_p defined by [3.15], their results may, in fact, be applied to axisymmetric bodies of any shape. This can be done by observing that $r_p = r_e$ for spheroids. Hence, if we merely reinterpret their r_p as being r_e for a general axisymmetric particle, then—in view of [2.29] and [2.30], relating r_e to B—they have, in fact, actually solved the general system of equations [8.19]–[8.21] for these more general bodies.

The three goniometrical factors $\langle \sin^2 \theta \sin 2\phi \rangle$, $\langle \sin^2 \theta \cos 2\phi \rangle$ and $\langle \sin^2 \theta \rangle$ (hereafter denoted by β, γ and δ, respectively) as functions of r_e and P, required in our rheological calculations, are not given explicitly by Scheraga *et al.* Rather, they are contained implicitly in their streaming birefringence (Scheraga *et al.* 1951) and intrinsic viscosity (Scheraga 1955) calculations, from which they may be extracted by means of the procedure described below.

For the phenomenon of streaming birefringence, Scheraga *et al.* (1951)* tabulate values of the extinction angle $\chi = \chi(r_e, P)$ and the "orientation factor" $F = F(r_e, P)$. These two parameters are related to the goniometrical factors β and γ via the relations

$$\frac{\pi}{4} - \chi = \frac{1}{2}\tan^{-1}\frac{\gamma}{\beta},$$

or, equivalently,

$$-\tan 2\chi = \beta/\gamma, \qquad [8.30]$$

and

$$\Delta n = 2\pi c\bar{n}^{-1}(g_1 - g_2)F,$$

in which

$$F = (\gamma^2 + \beta^2)^{1/2} \qquad [8.31]$$

is the orientation factor. Here, $(\pi/4) - \chi$ is the angle between the isocline and the principal strain axis, Δn is the difference in index of refraction along the two principal axes of the index of refraction tensor, c the volume concentration of solute particles, $\bar{n}$ the mean index of refraction of the solution, and $g_1 - g_2$ the optical anisotropy factor for the solute particles.

For $\infty > r_e \geq 1$ (i.e. $1 \geq B \geq 0$) it can be shown that $\gamma < 0$ and $\beta > 0$ for all P. Hence, from [8.30] and [8.31] we find for this case that

$$\langle \sin^2\theta \sin 2\phi \rangle = F \sin 2\chi, \qquad [8.32a]$$

$$\langle \sin^2\theta \cos 2\phi \rangle = -F \cos 2\chi, \qquad [8.32b]$$

valid for $r_e \geq 1$. Using the tabulated results of Scheraga *et al.* (1951) for this case, giving F and χ as functions of r_e and P, we have calculated values of the two goniometrical factors from [8.32]. These are tabulated in tables 6a and 6b as functions of r_e and P, for $\infty > r_e \geq 1$. Values of these two goniometrical factors for $0 < r_e \leq 1$ (i.e. $-1 \leq B \leq 0$) may be derived from this tabulation via the transformations indicated in [9.4].

Values of $\langle \sin^2\theta \sin 2\phi \rangle$ and $\langle \sin^2\theta \cos 2\phi \rangle$ at $r_e = \infty$ (i.e. $B = 1$) may be compared with those tabulated in table 5, derived from the independent calculations of Stewart & Sørensen (1972) in connection with the rheological properties of dumbbell suspensions. In general, the agreement is excellent for $P \leq 60$, thereby providing strong confirmation of the accuracy of both sets of numerics. As indicated by Scheraga *et al.* (1951), the values of χ and F for $P > 60$ are of uncertain validity, whence this same uncertainty attaches to the values of the goniometrical factors in tables 6a and 6b beyond $P = 60$.

* The quantities here designated by the symbols r_e, P, F and B are denoted in Scherega *et al.* (1951) as p, α, f and R, respectively.

Table 6a. Goniometric factors for a simple simple shear flow.
$\langle \sin^2\theta \sin 2\phi \rangle$*†

P \ r_e	1	2	3	4	5	7	10	16	25	50	∞
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	0.0000	0.0100	0.0133	0.0147	0.0159	0.0160	0.0163	0.0165	0.0166	0.0166	0.0167
0.50	0.0000	0.0199	0.0265	0.0292	0.0305	0.0318	0.0324	0.0328	0.0330	0.0331	0.0331
0.75	0.0000	0.0295	0.0393	0.0434	0.0454	0.0472	0.0482	0.0488	0.0490	0.0491	0.0491
1.00	0.0000	0.0389	0.0518	0.0571	0.0597	0.0621	0.0634	0.0642	0.0645	0.0646	0.0647
1.25	0.0000	0.0479	0.0637	0.0703	0.0735	0.0765	0.0781	0.0790	0.0794	0.0796	0.0796
1.50	0.0000	0.0564	0.0751	0.0828	0.0866	0.0900	0.0919	0.0930	0.0935	0.0937	0.0938
1.75	0.0000	0.0644	0.0858	0.0946	0.0989	0.1028	0.1050	0.1062	0.1067	0.1069	0.1071
2.00	0.0000	0.0719	0.0957	0.1055	0.1104	0.1147	0.1170	0.1185	0.1190	0.1193	0.1194
2.25	0.0000	0.0788	0.1049	0.1156	0.1209	0.1257	0.1283	0.1299	0.1304	0.1308	0.1309
2.50	0.0000	0.0851	0.1133	0.1249	0.1307	0.1358	0.1386	0.1403	0.1409	0.1413	0.1414
3.00	0.0000	0.0960	0.1279	0.1411	0.1476	0.1534	0.1566	0.1586	0.1593	0.1597	0.1598
3.50	0.0000	0.1046	0.1398	0.1542	0.1614	0.1678	0.1714	0.1735	0.1743	0.1748	0.1749
4.00	0.0000	0.1114	0.1491	0.1647	0.1724	0.1794	0.1833	0.1855	0.1864	0.1869	0.1871
4.50	0.0000	0.1164	0.1563	0.1729	0.1811	0.1887	0.1927	0.1952	0.1962	0.1967	0.1968
5.00	0.0000	0.1200	0.1617	0.1792	0.1880	0.1960	0.2003	0.2029	0.2040	0.2044	0.2047
6.00	0.0000	0.1237	0.1684	0.1876	0.1972	0.2060	0.2108	0.2137	0.2148	0.2154	0.2156
7.00	0.0000	0.1244	0.1714	0.1918	0.2022	0.2118	0.2170	0.2201	0.2214	0.2220	0.2223
8.00	0.0000	0.1232	0.1719	0.1936	0.2046	0.2149	0.2206	0.2240	0.2252	0.2260	0.2262
9.00	0.0000	0.1208	0.1709	0.1936	0.2053	0.2161	0.2222	0.2259	0.2272	0.2280	0.2282
10.00	0.0000	0.1177	0.1688	0.1926	0.2049	0.2164	0.2227	0.2266	0.2281	0.2288	0.2291
12.50	0.0000	0.1089	0.1617	0.1874	0.2010	0.2138	0.2210	0.2255	0.2272	0.2280	0.2283
15.00	0.0000	0.1001	0.1535	0.1807	0.1953	0.2093	0.2173	0.2222	0.2240	0.2250	0.2270
17.50	0.0000	0.0919	0.1453	0.1736	0.1891	0.2043	0.2128	0.2182	0.2201	0.2211	0.2215
20.00	0.0000	0.0847	0.1375	0.1667	0.1831	0.1990	0.2082	0.2138	0.2161	0.2172	0.2176
22.50	0.0000	0.0783	0.1303	0.1601	0.1772	0.1940	0.2039	0.2097	0.2121	0.2132	0.2136
25.00	0.0000	0.0727	0.1238	0.1541	0.1730	0.1895	0.1998	0.2062	0.2087	0.2100	0.2105
30.00	0.0000	0.0633	0.1121	0.1431	0.1619	0.1809	0.1922	0.1995	0.2023	0.2037	0.2041
35.00	0.0000	0.0560	0.1023	0.1336	0.1531	0.1734	0.1857	0.1935	0.1965	0.1981	0.1987
40.00	0.0000	0.0500	0.0939	0.1250	0.1452	0.1666	0.1798	0.1884	0.1917	0.1934	0.1940
45.00	0.0000	0.0452	0.0867	0.1174	0.1378	0.1602	0.1743	0.1834	0.1869	0.1889	0.1893
50.00	0.0000	0.0418	0.0803	0.1105	0.1310	0.1543	0.1691	0.1787	0.1826	0.1848	0.1852
60.00	0.0000	0.0348	0.0698	0.0986	0.1191	0.1430	0.1588	0.1692	0.1735	0.1758	0.1765
80.00	0.0000	0.0265	0.0551	0.0803	0.0993	0.1227	0.1388	0.1498	0.1541	0.1566	0.1576
100.00	0.0000	0.0214	0.0451	0.0609	0.0841	0.1054	0.1206	0.1309	0.1353	0.1376	0.1384
200.00	0.0000	0.0108	0.0235	0.0358	0.0458	0.0592	0.0687	0.0754	0.0782	0.0798	0.0803

* Tabulated values for $P > 60$ are of uncertain validity.

† Values of $\langle \sin^2\theta \sin 2\phi \rangle$ for r_e in the range $0 \le r_e \le 1$ may be derived from this tabulation by replacing r_e by $1/r_e$ and changing the algebraic sign of $\langle \sin^2\theta \sin 2\phi \rangle$ from positive to negative.

Table 6b. Goniometric factors for a simple shear flow.
$-\langle \sin^2\theta \cos 2\phi \rangle$ *†

P \ r_e	1	2	3	4	5	7	10	16	25	50	∞
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	0.0000	0.0004	0.0006	0.0006	0.0006	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007
0.50	0.0000	0.0017	0.0022	0.0025	0.0026	0.0027	0.0027	0.0028	0.0028	0.0028	0.0028
0.75	0.0000	0.0037	0.0049	0.0054	0.0057	0.0059	0.0060	0.0061	0.0061	0.0061	0.0061
1.00	0.0000	0.0065	0.0086	0.0095	0.0099	0.0103	0.0105	0.0106	0.0107	0.0107	0.0107
1.25	0.0000	0.0099	0.0132	0.0145	0.0152	0.0158	0.0161	0.0163	0.0164	0.0164	0.0164
1.50	0.0000	0.0140	0.0186	0.0205	0.0214	0.0222	0.0227	0.0229	0.0231	0.0231	0.0231
1.75	0.0000	0.0187	0.0247	0.0272	0.0284	0.0295	0.0301	0.0304	0.0306	0.0306	0.0307
2.00	0.0000	0.0237	0.0314	0.0345	0.0360	0.0374	0.0381	0.0386	0.0387	0.0388	0.0389
2.25	0.0000	0.0292	0.0386	0.0423	0.0442	0.0459	0.0468	0.0473	0.0475	0.0476	0.0476
2.50	0.0000	0.0350	0.0461	0.0506	0.0527	0.0547	0.0558	0.0564	0.0567	0.0568	0.0568
3.00	0.0000	0.0471	0.0619	0.0678	0.0706	0.0732	0.0746	0.0754	0.0757	0.0759	0.0759
3.50	0.0000	0.0595	0.0781	0.0853	0.0889	0.0920	0.0938	0.0948	0.0951	0.0954	0.0952
4.00	0.0000	0.0720	0.0941	0.1027	0.1070	0.1106	0.1127	0.1139	0.1143	0.1145	0.1146
4.50	0.0000	0.0841	0.1097	0.1197	0.1245	0.1287	0.1311	0.1324	0.1329	0.1332	0.1333
5.00	0.0000	0.0956	0.1245	0.1359	0.1412	0.1461	0.1486	0.1501	0.1507	0.1511	0.1511
6.00	0.0000	0.1166	0.1519	0.1655	0.1720	0.1778	0.1809	0.1828	0.1835	0.1838	0.1840
7.00	0.0000	0.1349	0.1759	0.1917	0.1993	0.2059	0.2096	0.2117	0.2125	0.2129	0.2130
8.00	0.0000	0.1505	0.1968	0.2147	0.2232	0.2307	0.2349	0.2372	0.2382	0.2386	0.2387
9.00	0.0000	0.1638	0.2149	0.2349	0.2443	0.2528	0.2572	0.2598	0.2609	0.2614	0.2616
10.00	0.0000	0.1752	0.2309	0.2526	0.2630	0.2722	0.2772	0.2801	0.2812	0.2818	0.2819
12.50	0.0000	0.1972	0.2628	0.2890	0.3016	0.3127	0.3188	0.3223	0.3235	0.3242	0.3246
15.00	0.0000	0.2127	0.2869	0.3170	0.3316	0.3446	0.3516	0.3556	0.3571	0.3579	0.3607
17.50	0.0000	0.2240	0.3055	0.3392	0.3557	0.3703	0.3783	0.3828	0.3846	0.3855	0.3858
20.00	0.0000	0.2323	0.3202	0.3572	0.3754	0.3916	0.4004	0.4056	0.4074	0.4085	0.4088
22.50	0.0000	0.2388	0.3322	0.3722	0.3920	0.4097	0.4191	0.4247	0.4268	0.4280	0.4283
25.00	0.0000	0.2438	0.3420	0.3850	0.4091	0.4256	0.4360	0.4421	0.4444	0.4456	0.4460
30.00	0.0000	0.2510	0.3572	0.4054	0.4297	0.4518	0.4637	0.4708	0.4734	0.4748	0.4753
35.00	0.0000	0.2558	0.3684	0.4211	0.4482	0.4729	0.4863	0.4943	0.4973	0.4988	0.4993
40.00	0.0000	0.2592	0.3768	0.4336	0.4632	0.4906	0.5056	0.5144	0.5177	0.5195	0.5201
45.00	0.0000	0.2617	0.3833	0.4436	0.4756	0.5056	0.5222	0.5319	0.5356	0.5376	0.5383
50.00	0.0000	0.2635	0.3885	0.4520	0.4864	0.5188	0.5370	0.5479	0.5520	0.5541	0.5549
60.00	0.0000	0.2660	0.3960	0.4647	0.5030	0.5404	0.5618	0.5747	0.5797	0.5823	0.5833
80.00	0.0000	0.2686	0.4045	0.4801	0.5245	0.5695	0.5966	0.6134	0.6200	0.6235	0.6246
100.00	0.0000	0.2699	0.4089	0.4887	0.5369	0.5872	0.6181	0.6378	0.6456	0.6498	0.6512
200.00	0.0000	0.2716	0.4154	0.5021	0.5569	0.6171	0.6556	0.6808	0.6910	0.6964	0.6983

* Tabulated values for $P > 60$ are of uncertain validity.
† Values of $\langle \sin^2\theta \cos 2\phi \rangle$ for r_e in the range $0 \leq r_e \leq 1$ may be derived from this tabulation by replacing r_e by $1/r_e$ and changing the algebraic sign of $\langle \sin^2\theta \cos 2\phi \rangle$ from negative to positive.

Table 6c. Goniometric factors for a simple shear flow.
$\langle \sin^2\theta \rangle$*

P \ r_e	1	2	3	4	5	7	10	16	25	50
0.00	0.6667	0.6667	0.6667	0.6667	0.6667	0.6667	0.6667	0.6667	0.6667	0.6667
0.25	0.6667	0.6667	0.6667	0.6667	0.6668	0.6668	0.6668	0.6668	0.6668	0.6668
0.50	0.6667	0.6668	0.6669	0.6670	0.6670	0.6670	0.6670	0.6671	0.6671	0.6671
0.75	0.6667	0.6670	0.6672	0.6673	0.6674	0.6675	0.6675	0.6675	0.6676	0.6676
1.00	0.6667	0.6672	0.6677	0.6679	0.6680	0.6681	0.6682	0.6682	0.6682	0.6683
1.25	0.6667	0.6675	0.6683	0.6686	0.6687	0.6688	0.6689	0.6690	0.6690	0.6690
1.50	0.6667	0.6680	0.6689	0.6692	0.6695	0.6698	0.6701	0.6701	0.6701	0.6701
1.75	0.6667	0.6682	0.6694	0.6701	0.6704	0.6708	0.6706	0.6709	0.6710	0.6710
2.00	0.6667	0.6684	0.6703	0.6710	0.6714	0.6719	0.6721	0.6722	0.6722	0.6722
2.25	0.6667	0.6693	0.6714	0.6720	0.6725	0.6730	0.6736	0.6738	0.6738	0.6738
2.50	0.6667	0.6695	0.6719	0.6732	0.6737	0.6743	0.6745	0.6749	0.6750	0.6750
3.00	0.6667	0.6708	0.6740	0.6754	0.6763	0.6771	0.6778	0.6778	0.6779	0.6779
3.50	0.6667	0.6717	0.6760	0.6777	0.6788	0.6798	0.6805	0.6806	0.6808	0.6808
4.00	0.6667	0.6725	0.6778	0.6800	0.6815	0.6826	0.6833	0.6838	0.6838	0.6839
4.50	0.6667	0.6738	0.6800	0.6826	0.6841	0.6854	0.6860	0.6867	0.6869	0.6870
5.00	0.6667	0.6753	0.6815	0.6849	0.6866	0.6882	0.6894	0.6895	0.6899	0.6900
6.00	0.6667	0.6766	0.6853	0.6893	0.6914	0.6934	0.6943	0.6953	0.6955	0.6956
7.00	0.6667	0.6790	0.6886	0.6932	0.6959	0.6982	0.6995	0.7006	0.7008	0.7008
8.00	0.6667	0.6799	0.6916	0.6970	0.7000	0.7027	0.7044	0.7053	0.7058	0.7060
9.00	0.6667	0.6816	0.6941	0.7003	0.7036	0.7070	0.7088	0.7097	0.7103	0.7106
10.00	0.6667	0.6827	0.6966	0.7032	0.7071	0.7108	0.7128	0.7140	0.7145	0.7148
12.50	0.6667	0.6848	0.7012	0.7098	0.7147	0.7191	0.7218	0.7233	0.7239	0.7243
15.00	0.6667	0.6862	0.7050	0.7146	0.7196	0.7246	0.7273	0.7288	0.7294	0.7330
17.50	0.6667	0.6879	0.7087	0.7207	0.7262	0.7323	0.7359	0.7378	0.7387	0.7393
20.00	0.6667	0.6888	0.7108	0.7236	0.7307	0.7374	0.7415	0.7441	0.7447	0.7451
22.50	0.6667	0.6892	0.7133	0.7269	0.7346	0.7422	0.7465	0.7492	0.7501	0.7506
25.00	0.6667	0.6897	0.7149	0.7297	0.7408	0.7465	0.7513	0.7542	0.7554	0.7559
30.00	0.6667	0.6910	0.7177	0.7344	0.7441	0.7537	0.7593	0.7627	0.7642	0.7647
35.00	0.6667	0.6912	0.7196	0.7380	0.7488	0.7597	0.7661	0.7700	0.7716	0.7738
40.00	0.6667	0.6915	0.7215	0.7410	0.7527	0.7649	0.7722	0.7768	0.7784	0.7793
45.00	0.6667	0.6917	0.7226	0.7432	0.7559	0.7692	0.7772	0.7821	0.7841	0.7852
50.00	0.6667	0.6920	0.7236	0.7452	0.7587	0.7729	0.7817	0.7873	0.7895	0.7905
60.00	0.6667	0.6922	0.7250	0.7480	0.7627	0.7790	0.7890	0.7954	0.7980	0.7993

* These same values of $\langle \sin^2\theta \rangle$ apply if r_e is replaced by $1/r_e$.

Numerical values of the remaining goniometrical factor $\langle \sin^2\theta\rangle$ as a function of r_e and P (for $\infty > r_e \geq 1$) can be obtained from Scheraga's (1955) tabulation of the intrinsic viscosity* of prolate spheroids as a function of these parameters, in conjunction with tables 6a and 6b. In particular, from [8.5] we have that

$$\langle \sin^2\theta\rangle = \frac{4Q_1}{3Q_2} + \frac{6}{BP}\left(1+\frac{4Q_3}{3Q_2}\right)\langle \sin^2\theta \sin 2\phi\rangle - \frac{1}{B}\left(1+\frac{4Q_3^o}{3Q_2}\right)\langle \sin^2\theta\cos 2\phi\rangle - \frac{4[\eta]}{15Q_2}. \quad [8.33]$$

For $r_e \geq 1$, values of $\langle \sin^2\theta\sin 2\phi\rangle$ and $\langle \sin^2\theta\cos 2\phi\rangle$ appearing on the right-hand side of this relation are available as functions of r_e and P in tables 6a and 6b. Values of Q_1, Q_2, Q_3 and Q_3^o and B as functions of r_e may be obtained for prolate spheroids (for which $r_e = r_p$) from [3.8]–[3.15]. Equation [8.33] then enables $\langle \sin^2\theta\rangle$ to be calculated in terms of r_e and P. The results of such a calculation are tabulated in table 6c. In view of the transformation indicated in [9.4] these same values of $\langle \sin^2\theta\rangle$ apply if r_e is replaced by $1/r_e$ in the table. Hence, this furnishes the goniometric factor $\langle \sin^2\theta\rangle$ over the complete range of equivalent axis ratios, $0 \leq r_e < \infty$, at least for $|B| < 1$. An independent check of the validity of the transformation [9.4] was made by a comparable calculation of $\langle \sin^2\theta\rangle$ using Scheraga's (1955) tabulation of the intrinsic viscosity of *oblate* spheroids ($0 \leq r_e \leq 1$). Results substantially identical to those tabulated in table 6c were obtained by this procedure, generally to within a few units in the last significant figure.

The values of $\langle \sin^2\theta\rangle$ in table 6c at the larger values of r_e ought to be comparable to those in table 5 at $B = 1$ ($r_e = \infty$), which derive from the dumbbell analysis of Stewart & Sørensen (1972). In general, the agreement between the two independent sets of $\langle \sin^2\theta\rangle$ is quite good, again providing confirmation of the numerics of both sets of authors.

Yet further support for the general accuracy of the values tabulated in tables 6a, 6b and 6c derives from the asymptotic analyses of Hinch & Leal (1972) for $P \gg 1$, discussed in Section 9.

Values of the goniometric factors in tables 6a, 6b and 6c may be utilized to compute the variation of intrinsic viscosity and normal stresses with rotary Péclet number for any axisymmetric particles immersed in a simple shear flow. This requires use of [8.5]–[8.7] in conjunction with the Q and B values for any of the bodies characterized in Section 3. In this manner, numerical values of the intrinsic viscosity function $[\eta]$, and of the dimensionless normal stresses. $\Sigma_i \equiv [\tau_i]P \quad (i = 1, 2)$,

$$\Sigma_1 = \tau_1/\phi\mu_o D_r, \qquad \Sigma_2 = \tau_2/\phi\mu_o D_r, \qquad [8.34a, b]$$

were calculated as functions of the dimensionless shear rate $P = G/D_r$ for both prolate and oblate spheroids of various aspect ratios r_p, the results being cited in tables 7a, 7b, 7c and 8a, 8b, 8c. Results for the intrinsic viscosity agree, of course, with those of Scheraga (1955),

* In Scheraga's (1955) notation, the intrinsic viscosity is denoted by ν, rather than $[\eta]$, as in the present paper. Moreover, his spheroidal integrals, J, K, L, M and N, are related to our Q values by means of the following relations:

$$J + K = 5(Q_1 - \tfrac{3}{4}Q_2), \qquad L = 5(Q_1 + Q_3^o), \qquad M = 5Q_1, \qquad N = 30B^{-1}(Q_3 - Q_3^o).$$

These relations, in conjunction with [8.10b], show that Scheraga's (1955) general expression for the intrinsic viscosity of a dilute suspension of spheroids (cf. his equation [9]) is identical with our equation [8.5].

Table 7a. Intrinsic viscosity for prolate spheroids in a simple shear flow.

$[\eta]$

P \ r_p	1	2	3	4	5	7	10	16	25	50
0.00	2.500	2.908	3.685	4.663	5.806	8.533	13.63	27.18	55.19	176.8
0.25	2.500	2.907	3.683	4.661	5.802	8.526	13.62	27.15	55.13	176.6
0.50	2.500	2.906	3.679	4.653	5.791	8.506	13.59	27.07	54.96	176.0
0.75	2.500	2.903	3.672	4.641	5.773	8.474	13.53	26.94	54.68	175.1
1.00	2.500	2.899	3.663	4.624	5.748	8.429	13.45	26.76	54.29	173.8
1.25	2.500	2.895	3.651	4.604	5.717	8.374	13.34	26.54	53.81	172.1
1.50	2.500	2.890	3.637	4.579	5.681	8.310	13.23	26.28	53.25	170.3
1.75	2.500	2.884	3.621	4.552	5.640	8.237	13.09	25.98	52.62	168.1
2.00	2.500	2.877	3.604	4.522	5.596	8.158	12.95	25.66	51.93	165.8
2.25	2.500	2.871	3.586	4.490	5.548	8.074	12.80	25.32	51.20	163.4
2.50	2.500	2.863	3.566	4.457	5.499	7.986	12.64	24.97	50.44	160.8
3.00	2.500	2.848	3.526	4.387	5.396	7.804	12.31	24.24	48.86	155.5
3.50	2.500	2.832	3.485	4.316	5.291	7.619	11.97	23.49	47.26	150.1
4.00	2.500	2.816	3.444	4.246	5.188	7.437	11.64	22.77	45.70	144.9
4.50	2.500	2.801	3.405	4.179	5.089	7.263	11.32	22.07	44.21	139.9
5.00	2.500	2.787	3.367	4.115	4.995	7.097	11.03	21.41	42.80	135.2
6.00	2.500	2.760	3.299	3.999	4.824	6.797	10.48	20.22	40.24	126.6
7.00	2.500	2.738	3.240	3.897	4.675	6.537	10.01	19.19	38.04	119.2
8.00	2.500	2.718	3.189	3.810	4.547	6.312	9.608	18.30	36.14	112.9
9.00	2.500	2.702	3.145	3.734	4.435	6.117	9.257	17.53	34.50	107.5
10.00	2.500	2.688	3.107	3.668	4.338	5.947	8.950	16.86	33.07	102.7
12.50	2.500	2.661	3.037	3.536	4.143	5.603	8.332	15.51	30.21	93.19
15.00	2.500	2.642	2.995	3.435	3.989	5.329	7.833	14.42	27.89	85.54
17.50	2.500	2.629	2.934	3.364	3.880	5.139	7.496	13.69	26.34	80.43
20.00	2.500	2.619	2.900	3.300	3.788	4.974	7.199	13.05	24.98	75.93
22.50	2.500	2.611	2.874	3.250	3.712	4.839	6.956	12.52	23.86	72.28
25.00	2.500	2.605	2.852	3.208	3.647	4.723	6.745	12.06	22.90	69.12
30.00	2.500	2.597	2.819	3.142	3.545	4.536	6.406	11.32	21.35	64.05
35.00	2.500	2.591	2.795	3.092	3.465	4.389	6.138	10.74	20.12	60.39
40.00	2.500	2.587	2.777	3.053	3.401	4.269	5.918	10.27	19.12	56.80
45.00	2.500	2.584	2.763	3.021	3.348	4.167	5.728	9.847	18.24	53.94
50.00	2.500	2.582	2.752	2.995	3.303	4.078	5.560	9.477	17.46	51.40
60.00	2.500	2.579	2.736	2.955	3.232	3.933	5.278	8.845	16.12	47.04

Table 7b. Normal stress function for prolate spheroids in a simple shear flow.

$-\Sigma_1$

P \ r_p	1	2	3	4	5	7	10	16	25	50
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	0.0000	0.0007	0.0016	0.0026	0.0037	0.0066	0.0124	0.0295	0.0680	0.2494
0.50	0.0000	0.0027	0.0062	0.0102	0.0148	0.0263	0.0496	0.1178	0.2715	0.9984
0.75	0.0000	0.0061	0.0141	0.0230	0.0333	0.0591	0.1115	0.2648	0.6106	2.247
1.00	0.0000	0.0107	0.0249	0.0407	0.0590	0.1048	0.1978	0.4708	1.085	3.995
1.25	0.0000	0.0163	0.0362	0.0598	0.0881	0.1581	0.3260	0.6672	1.594	6.575
1.50	0.0000	0.0223	0.0523	0.0870	0.1240	0.2157	0.3889	0.9400	2.222	7.828
1.75	0.0000	0.0301	0.0710	0.1142	0.1659	0.2899	0.5762	1.308	2.905	11.03
2.00	0.0000	0.0404	0.0892	0.1450	0.2094	0.3680	0.6878	1.639	3.719	13.94
2.25	0.0000	0.0463	0.1082	0.1642	0.2569	0.4209	0.8223	2.027	4.519	16.56
2.50	0.0000	0.0580	0.1322	0.2102	0.3057	0.5357	0.9815	2.365	5.356	19.99
3.00	0.0000	0.0771	0.1761	0.2841	0.4068	0.7098	1.301	3.124	7.218	26.55
3.50	0.0000	0.0983	0.2222	0.3587	0.5113	0.8911	1.659	3.946	9.051	33.54
4.00	0.0000	0.1200	0.2696	0.4328	0.6136	1.071	1.991	4.651	10.77	39.58
4.50	0.0000	0.1399	0.3136	0.5013	0.7123	1.238	2.318	5.403	12.36	45.43
5.00	0.0000	0.1580	0.3600	0.5690	0.8085	1.399	2.560	6.092	13.93	50.55
6.00	0.0000	0.1962	0.4398	0.6948	0.9828	1.691	3.130	7.264	16.65	61.16
7.00	0.0000	0.2261	0.5117	0.8092	1.136	1.941	3.571	8.259	18.90	69.68
8.00	0.0000	0.2560	0.5760	0.9072	1.270	2.158	3.928	9.131	20.79	76.36
9.00	0.0000	0.2709	0.6300	0.9954	1.391	2.340	4.251	9.852	22.38	81.43
10.00	0.0000	0.2990	0.6850	1.076	1.497	2.503	4.526	10.43	23.74	86.60
12.50	0.0000	0.3425	0.7950	1.241	1.714	2.844	5.066	11.59	26.21	95.67
15.00	0.0000	0.3750	0.8805	1.382	1.917	3.170	5.654	12.90	29.18	115.4
17.50	0.0000	0.3955	0.9451	1.468	2.034	3.306	5.777	13.00	29.21	106.0
20.00	0.0000	0.4140	1.006	1.574	2.156	3.400	6.030	13.39	30.10	109.1
22.50	0.0000	0.4297	1.053	1.656	2.261	3.620	6.230	13.74	30.74	111.0
25.00	0.0000	0.4400	1.098	1.728	2.323	3.523	6.388	14.00	31.11	112.4
30.00	0.0000	0.4557	1.164	1.848	2.520	3.963	6.660	14.40	31.73	114.1
35.00	0.0000	0.4690	1.211	1.953	2.660	4.151	6.885	14.67	32.13	114.6
40.00	0.0000	0.4760	1.260	2.036	2.784	4.312	7.064	14.80	32.28	115.0
45.00	0.0000	0.4860	1.296	2.111	2.889	4.464	7.245	15.06	32.56	115.6
50.00	0.0000	0.4920	1.325	2.175	2.980	4.605	7.415	15.23	32.73	116.0
60.00	0.0000	0.4990	1.362	2.274	3.144	4.842	7.734	15.65	33.32	117.4

Table 7c. Normal stress function for prolate spheroids in a simple shear flow.

Σ_2

P \ r_p	1	2	3	4	5	7	10	16	25	50
0.00	0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	0.000	0.0061	0.0171	0.0304	0.0456	0.0814	0.1479	0.3242	0.6894	2.281
0.50	0.000	0.0242	0.0682	0.1211	0.1817	0.3242	0.5890	1.291	2.747	9.085
0.75	0.000	0.0540	0.1521	0.2704	0.4055	0.7237	1.315	2.884	6.135	20.30
1.00	0.000	0.0950	0.2673	0.4751	0.7128	1.272	2.313	5.074	10.79	35.74
1.25	0.000	0.1457	0.4106	0.7286	1.091	1.945	3.502	7.770	16.48	53.75
1.50	0.000	0.2065	0.5796	1.027	1.542	2.750	5.014	10.96	23.30	77.28
1.75	0.000	0.2751	0.7707	1.370	2.052	3.660	6.616	14.53	31.02	102.1
2.00	0.000	0.3486	0.9834	1.745	2.616	4.662	8.458	18.55	39.42	130.0
2.25	0.000	0.4331	1.214	2.150	3.222	5.801	10.45	22.81	48.59	160.7
2.50	0.000	0.5172	1.453	2.583	3.867	6.890	12.53	27.39	58.29	192.4
3.00	0.000	0.7014	1.969	3.350	5.236	9.329	16.95	37.08	78.72	260.0
3.50	0.000	0.8906	2.509	4.449	6.670	11.88	21.56	47.15	100.1	330.5
4.00	0.000	1.082	3.053	5.422	8.131	14.48	26.28	57.51	122.0	402.7
4.50	0.000	1.273	3.599	6.398	9.592	17.10	30.99	67.83	144.0	475.2
5.00	0.000	1.458	4.129	7.358	11.04	19.69	35.74	78.08	165.8	547.2
6.00	0.000	1.799	5.151	9.214	13.85	24.72	44.85	98.11	208.1	685.6
7.00	0.000	2.112	6.104	10.96	16.51	29.53	53.60	117.2	248.6	818.1
8.00	0.000	2.386	6.981	12.61	19.03	34.10	61.99	135.5	287.2	945.1
9.00	0.000	2.630	7.791	14.14	21.41	38.44	69.92	152.3	324.0	1066
10.00	0.000	2.844	8,537	15.59	23.67	42.61	77.54	169.6	359.3	1181
12.50	0.000	3.283	10.18	18.86	28.83	52.19	95.26	208.6	441.7	1449
15.00	0.000	3.615	11.55	21.71	33.40	60.82	111.3	243.8	515.7	1691
17.50	0.000	3.869	12.73	24.26	37.59	68.92	126.5	277.5	587.1	1923
20.00	0.000	4,070	13.73	26.52	41.42	76.33	140.5	308.6	653.2	2138
22.50	0.000	4.230	14.61	28.57	44.93	83.33	153.9	338.2	716.0	2342
25.00	0.000	4.360	15.39	30.47	48.63	90.07	166.8	367.4	777.7	2543
30.00	0.000	4.551	16.68	33.81	54.26	102.5	191.0	422.2	894.0	2922
35.00	0.000	4.694	17.84	36.70	59.60	113.9	213.8	473.9	1004	3283
40.00	0.000	4.788	18.55	39.13	64.36	124.5	235.2	523.7	1111	3626
45.00	0.000	4.865	19.24	41.25	68.51	134.2	255.3	570.1	1210	3954
50.00	0.000	4.990	19.78	43.06	72.21	143.1	274.1	614.3	1306	4269
60.00	0.000	4.992	20.60	45.98	78.47	158.4	306.9	692.7	1476	4825

Table 8a. Intrinsic viscosity for oblate spheroids in a simple shear flow.

$[\eta]$

P \ r_p^{-1}	1	2	3	4	5	7	10	16	25	50
0.00	2.500	2.854	3.431	4.059	4.708	6.032	8.043	12.10	18.19	35.16
0.25	2.500	2.854	3.430	4.058	4.706	6.029	8.039	12.09	18.18	35.13
0.50	2.500	2.853	3.427	4.053	4.700	6.019	8.025	12.07	18.15	35.06
0.75	2.500	2.850	3.422	4,045	4.689	6.004	8.002	12.03	18.09	34.95
1.00	2.500	2.847	3.415	4.034	4.675	5.983	7.971	11.98	18.01	34.79
1.25	2.500	2.844	3.406	4.021	4.657	5.956	7.933	11.92	17.91	34.59
1.50	2.500	2.839	3.396	4.005	4.636	5.926	7.887	11.84	17.80	34.36
1.75	2.500	2.834	3.384	3.988	4.613	5.891	7.837	11.76	17.67	34.10
2.00	2.500	2.829	3.371	3.968	4.587	5.853	7.781	11.67	17.52	33.82
2.25	2.500	2.823	3.358	3.948	4.560	5.813	7.722	11.57	17.37	33.51
2.50	2.500	2.817	3.344	3.926	4.531	5.770	7.660	11.47	17.22	33.20
3.00	2.500	2.804	3.314	3.881	4.471	5.682	7.531	11.26	16.89	32.54
3.50	2.500	2.790	3.283	3.834	4.410	5.592	7.399	11.05	16.55	31.87
4.00	2.500	2.777	3.253	3.788	4.349	5.503	7.269	10.84	16.22	31.21
4.50	2.500	2.764	3.224	3.744	4.290	5.417	7.144	10.64	15.90	30.57
5.00	2.500	2.751	3.196	3.701	4.234	5.335	7.024	10.44	15.60	29.96
6.00	2.500	2.729	3.144	3.623	4.131	5.184	6.803	10.08	15.04	28.84
7.00	2.500	2.709	3.099	3.555	4.040	5.050	6.608	9.769	14.54	27.85
8.00	2.500	2.692	3.060	3.494	3.960	4.933	6.436	9.490	14.11	26.97
9.00	2.500	2.678	3.026	3.442	3.890	4.829	6.284	9.244	13.72	26.20
10.00	2.500	2.666	2.996	3.395	3.827	4.737	6.149	9.025	13.38	25.51
12.50	2.500	2.642	2.936	3.300	3.699	4.546	5.868	8.570	12.66	24.08
15.00	2.500	2.626	2.891	3.225	3.597	4.390	5.636	8.190	12.07	22.89
17.50	2.500	2.614	2.856	3.169	3.520	4.276	5.468	7.918	11.64	22.03
20.00	2.500	2.605	2.829	3.123	3.955	4.176	5.319	7.674	11.26	21.27
22.50	2.500	2.598	2.807	3.084	3.400	4.091	5.192	7.467	10.93	20.62
25.00	2.500	2.593	2.789	3.051	3.352	4.017	5.079	7.281	10.64	20.03
30.00	2.500	2.585	2.760	2.998	3.275	3.894	4.893	6.973	10.15	19.06
35.00	2.500	2.580	2.740	2.957	3.215	3.795	4.741	6.721	9.753	18.24
40.00	2.500	2.577	2.724	2.925	3.165	3.712	4.612	6.505	9.411	17.57
45.00	2.500	2.574	2.712	2.899	3.124	3.643	4.501	6.318	9.114	16.97
50.00	2.500	2.572	2.703	2.878	3.090	3.581	4.403	6.149	8.844	16.43
60.00	2.500	2.570	2.690	2.846	3.036	3.483	4.240	5.867	8.391	15.51

Table 8b. Normal stress function for oblate spheroids in a simple shear flow.
$-\Sigma_1$

P \ r_p^{-1}	1	2	3	4	5	7	10	16	25	50
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	0.0000	0.0010	0.0025	0.0041	0.0057	0.0088	0.0135	0.0226	0.0362	0.0742
0.50	0.0000	0.0039	0.0101	0.0164	0.0227	0.0352	0.0555	0.0902	0.1440	0.2959
0.75	0.0000	0.0087	0.0225	0.0367	0.0508	0.0788	0.1201	0.2019	0.3227	0.6620
1.00	0.0000	0.0153	0.0397	0.0647	0.0896	0.1389	0.2118	0.3560	0.5692	1.168
1.25	0.0000	0.0223	0.0571	0.0975	0.1361	0.2152	0.3198	0.5157	0.8624	1.781
1.50	0.0000	0.0337	0.0814	0.1396	0.1930	0.2979	0.4539	0.7729	1.224	2.493
1.75	0.0000	0.0444	0.1113	0.1834	0.2546	0.3976	0.6007	1.011	1.616	3.313
2.00	0.0000	0.0552	0.1434	0.2362	0.3266	0.5066	0.7668	1.286	2.056	4.204
2.25	0.0000	0.0684	0.1748	0.2882	0.3991	0.6205	0.9423	1.605	2.545	5.223
2.50	0.0000	0.0815	0.2095	0.3465	0.4805	0.7470	1.130	1.916	3.041	6.211
3.00	0.0000	0.1104	0.2850	0.4671	0.6483	1.006	1.525	2.586	4.104	8.402
3.50	0.0000	0.1421	0.3643	0.5967	0.8221	1.278	1.941	3.261	5.206	10.67
4.00	0.0000	0.1712	0.4424	0.7260	1.001	1.554	2.362	3.962	6.328	12.98
4.50	0.0000	0.2007	0.5188	0.8509	1.179	1.829	2.774	4.647	7.454	15.28
5.00	0.0000	0.2315	0.5950	0.9775	1.353	2.097	3.188	5.370	8.814	17.53
6.00	0.0000	0.2832	0.7404	1.218	1.687	2.617	3.984	6.719	10.69	21.91
7.00	0.0000	0.3318	0.8729	1.437	1.999	3.110	4.732	7.956	12.71	26.04
8.00	0.0000	0.3752	0.9928	1.647	2.292	3.568	5.442	9.158	14.62	29.61
9.00	0.0000	0.4113	1.103	1.836	2.563	3.999	6.212	10.29	16.42	33.68
10.00	0.0000	0.4430	1.204	2.017	2.823	4.410	6.738	11.35	18.13	37.19
12.50	0.0000	0.5100	1.424	2.413	3.399	5.341	8.186	13.81	22.07	45.29
15.00	0.0000	0.5565	1.604	2.760	3.912	6.197	9.540	16.13	25.51	52.94
17.50	0.0000	0.5932	1.755	3.050	4.349	6.918	10.67	18.06	28.87	59.27
20.00	0.0000	0.6240	1.882	3.310	4.754	7.604	11.77	19.94	31.91	65.49
22.50	0.0000	0.6457	1.991	3.544	5.126	8.244	12.81	21.72	34.77	71.35
25.00	0.0000	0.6650	2.088	3.760	5.478	8.853	13.79	23.45	37.57	77.15
30.00	0.0000	0.6930	2.250	4.134	6.093	9.963	15.62	26.68	42.80	87.90
35.00	0.0000	0.7105	2.373	4.456	6.633	10.98	17.33	29.69	47.66	98.11
40.00	0.0000	0.7240	2.476	4.720	7.116	11.91	18.92	32.56	52.35	107.7
45.00	0.0000	0.7335	2.561	4.955	7.533	12.75	20.40	35.24	56.71	116.9
50.00	0.0000	0.7400	2.620	5.150	7.900	13.54	21.78	37.76	60.91	125.6
60.00	0.0000	0.7500	2.718	5.460	8.520	14.85	24.19	42.23	68.32	141.1

Table 8c. Normal stress function for oblate spheroids in a simple shear flow.

Σ_2

P \ r_p^{-1}	1	2	3	4	5	7	10	16	25	50
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	0.0000	0.0041	0.0092	0.0138	0.0181	0.0263	0.0382	0.0613	0.0951	0.1903
0.50	0.0000	0.0164	0.0351	0.0549	0.0721	0.1047	0.1600	0.2437	0.3787	0.7574
0.75	0.0000	0.0365	0.0814	0.1224	0.1607	0.2335	0.3385	0.5436	0.8439	1.689
1.00	0.0000	0.0641	0.1430	0.2148	0.2821	0.4099	0.5942	0.9543	1.481	2.965
1.25	0.0000	0.0995	0.2226	0.3298	0.4332	0.6256	0.9141	1.489	2.276	4.542
1.50	0.0000	0.1386	0.3141	0.4656	0.6114	0.8883	1.288	2.056	3.209	6.432
1.75	0.0000	0.1849	0.4156	0.6230	0.8167	1.183	1.720	2.756	4.280	8.558
2.00	0.0000	0.2374	0.5280	0.7908	1.039	1.507	2.186	3.515	5.451	10.92
2.25	0.0000	0.2922	0.6527	0.9769	1.283	1.860	2.698	4.310	6.702	13.42
2.50	0.0000	0.3515	0.7832	1.172	1.538	2.030	3.236	5.177	8.052	16.15
3.00	0.0000	0.4752	1.059	1.588	2.083	3.020	4.382	7.014	10.91	21.85
3.50	0.0000	0.6016	1.347	2.020	2.657	3.849	5.586	8.965	13.91	27.86
4.00	0.0000	0.7336	1.641	2.463	3.238	4.698	6.816	10.94	16.98	33.98
4.50	0.0000	0.8617	1.935	2.909	3.822	5.554	8.056	12.95	20.07	40.18
5.00	0.0000	0.9850	2.222	3.346	4.403	6.406	9.290	14.90	22.90	46.34
6.00	0.0000	1.219	2.770	4.195	5.531	8.060	11.70	18.78	29.18	58.40
7.00	0.0000	1.429	3.284	4.998	6.605	9.644	14.02	22.53	34.99	70.03
8.00	0.0000	1.615	3.760	5.753	7.623	11.16	16.25	26.13	40.56	81.21
9.00	0.0000	1.781	4.199	6.466	8.591	12.61	18.22	29.58	45.93	91.96
10.00	0.0000	1.927	4.604	7.134	9.510	14.00	20.42	32.91	51.13	102.3
12.50	0.0000	2.225	5.498	8.651	11.62	17.23	25.23	40.77	63.37	126.8
15.00	0.0000	2.453	6.249	9.980	13.51	20.15	29.62	47.93	74.81	148.6
17.50	0.0000	2.625	6.890	11.17	15.24	22.93	33.82	54.87	85.38	171.0
20.00	0.0000	2.762	7.442	12.24	16.83	25.47	37.73	61.31	95.57	191.5
22.50	0.0000	2.871	7.927	13.21	18.30	27.88	41.48	67.51	105.3	211.1
25.00	0.0000	2.963	8.360	14.11	19.86	30.22	45.10	73.63	114.9	230.5
30.00	0.0000	3.093	9.069	15.70	22.22	34.54	51.92	85.25	133.3	267.5
35.00	0.0000	3.189	9.646	17.07	24.47	38.55	58.41	96.25	150.7	302.7
40.00	0.0000	3.256	10.11	18.24	26.49	42.27	64.53	106.9	167.7	337.2
45.00	0.0000	3.308	10.49	19.25	28.26	45.68	70.29	116.9	183.7	370.0
50.00	0.0000	3.350	10.80	20.12	29.83	48.84	75.70	126.5	199.2	401.7
60.00	0.0000	3.396	11.26	21.53	32.50	54.25	85.14	143.7	226.7	441.8

since the goniometric factors used in their computation were derived, in part, from Scheraga's (1955) tabulation of $[\eta]$ vs r_p and P. The normal stress results cited in the tables have not previously been available over the complete spectrum of Péclet numbers. Some of the results in these tables are also presented graphically in figures 7–12.

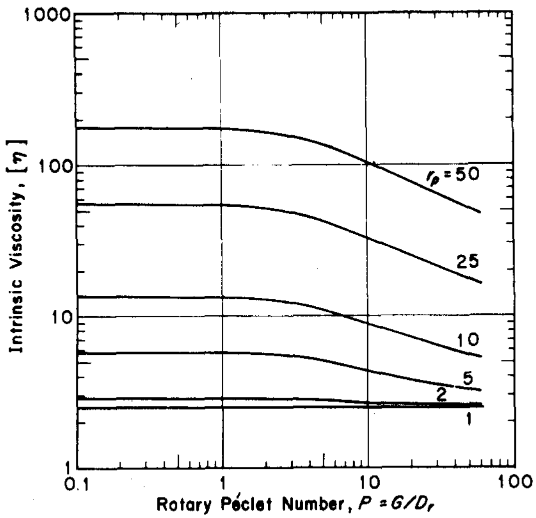

Figure 7. Variation of intrinsic viscosity with shear rate for prolate spheroids of various aspect ratios suspended in a simple shear flow.

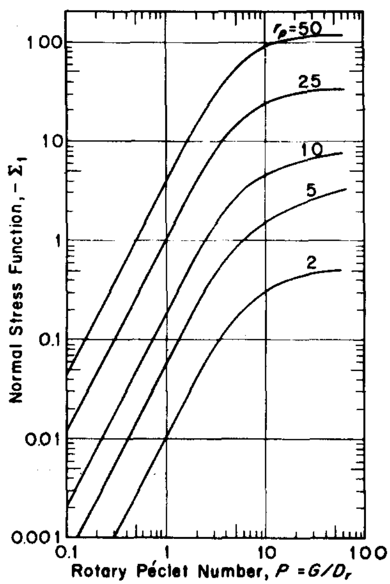

Figure 8. Variation of the first normal stress difference with shear rate for prolate spheroids of various aspect ratios suspended in a simple shear flow.

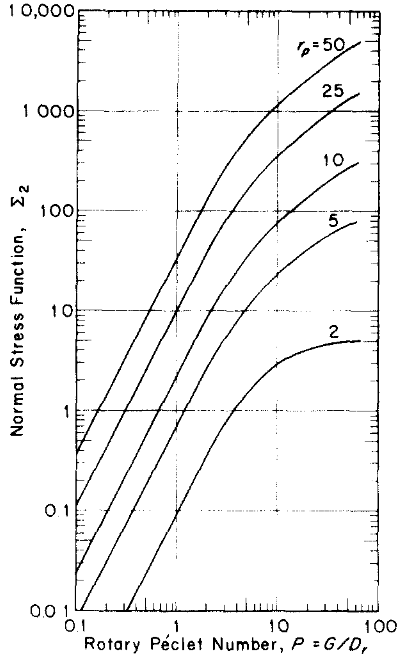

Figure 9. Variation of the second normal stress difference with shear rate for prolate spheroids of various aspect ratios suspended in a simple shear flow.

At sufficiently low rates of shear these suspensions display Newtonian behavior, in that the intrinsic viscosity, $[\eta]_0$ say, is sensibly independent of the shear rate, and the normal stresses are effectively zero. Since the intrinsic viscosity decreases monotonically with increasing shear rate, the general rheological behavior of these suspensions is of the shear-thinning type. As the rate of shear is increased indefinitely, the intrinsic viscosity ultimately approaches an asymptotic value, $[\eta]_\infty$ (see tables 9a and 9b).

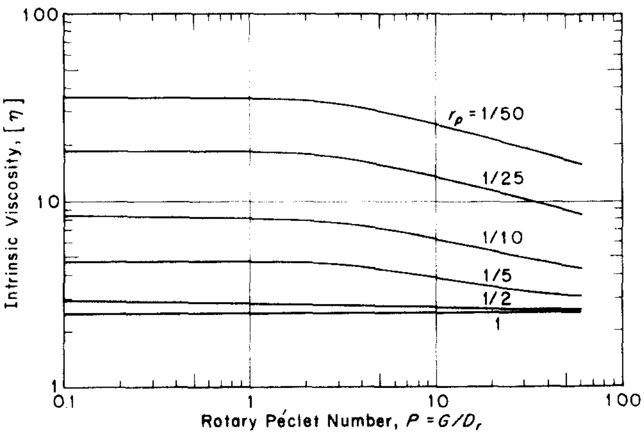

Figure 10. Variation of intrinsic viscosity with shear rate for oblate spheroids of various aspect ratios suspended in a simple shear flow.

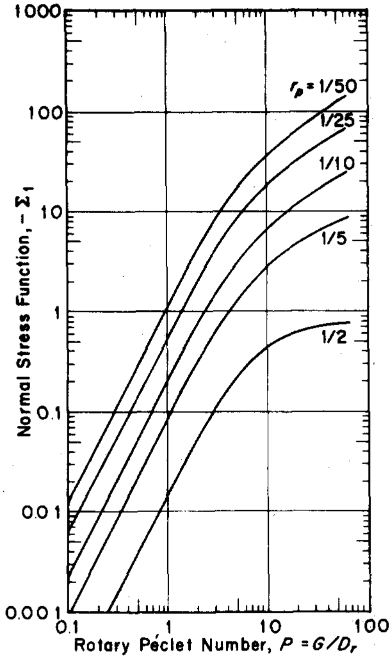

Figure 11. Variation of the first normal stress difference with shear rate for oblate spheroids of various aspect ratios suspended in a simple shear flow.

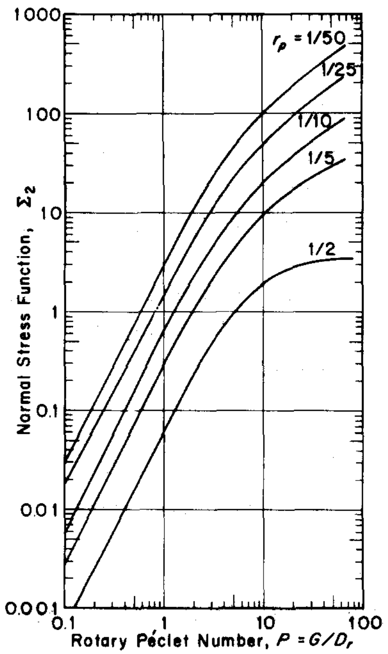

Figure 12. Variation of the second normal stress difference with shear rate for oblate spheroids of various aspect ratios suspended in a simple shear flow.

Table 9a. Limiting values of the three viscometric functions at infinite shear rate for prolate spheroids.

r_p	$[\eta]_\infty$	$-(\Sigma_1)_\infty$	$(\Sigma_2)_\infty$
1	2.5000	0.00000	0.00000
2	2.5695	0.51826	5.1612
3	2.6836	1.5523	24.178
4	2.8010	2.9354	66.092
5	2.9184	4.5850	142.06
7	3.1375	8.5585	449.21
10	3.4366	15.858	1539.6
16	3.9296	34.518	8026.6
25	4.6208	70.824	39,672
50	7.6823	212.24	497,980

Table 9b. Limiting values of the three viscometric functions at infinite shear rate for oblate spheroids.

r_p^{-1}	$[\eta]_\infty$	$-(\Sigma_1)_\infty$	$(\Sigma_2)_\infty$
1	2.5000	0.00000	0.0000
2	2.5659	0.76932	3.5249
3	2.6507	3.1300	13.296
4	2.7239	7.5432	31.339
5	2.7828	14.571	60.094
7	2.8675	38.804	159.69
10	2.9403	109.74	451.96
16	2.9977	437.76	1800.9
25	3.0287	1650.1	6759.3
50	3.1768	13,135	53,351

The primary and secondary normal stress differences are of opposite algebraic signs. The magnitudes of both increase monotonically from zero as the shear rate is increased, eventually attaining asymptotic values, $(\Sigma_1)_\infty$ and $(\Sigma_2)_\infty$, at an effectively infinite rate of shear (see tables 9a and 9b).

In general, the "exact" values of the viscometric functions cited in tables 7 and 8 at small Péclet numbers compare favorably with the asymptotic expressions that may be derived for these quantities from [8.13], namely

$$[\eta] = [\eta]_0 - F_\eta(r_e)P^2 + O(P^4) \qquad [8.35a]$$

$$\Sigma_1 = -F_1(r_e)P^2 + O(P^4) \qquad [8.35b]$$

$$\Sigma_2 = F_2(r_e)P^2 + O(P^4) \qquad [8.35c]$$

with $[\eta]_0$ given as a function of r_e by [7.8], and

$$F_\eta(r_e) = \tfrac{1}{1260}B^2(12Q_2 + 6Q_3 + 35B^{-1}N), \qquad [8.36a]$$

$$F_1(r_e) = B[\tfrac{1}{6}N - \tfrac{1}{7}(Q_3 - Q_2)], \qquad [8.36b]$$

$$F_2(r_e) = B[\tfrac{1}{6}N + \tfrac{1}{7}(Q_3 - Q_2)]. \qquad [8.36c]$$

These demonstrate that the viscosity η attains a constant limiting value at $P = 0$, whereas the normal stress differences τ_1 and τ_2 vanish in this limit.

In the opposite limit, where $P \gg 1$ (and, simultaneously, $P \gg r_e^3 + r_e^{-3}$), the goniometric factors required in [8.5]–[8.7] may be obtained from table 10 in the next section. The limiting values, $[\eta]_\infty$, $(\Sigma_1)_\infty$ and $(\Sigma_2)_\infty$, of the viscometric functions thereby obtained for spheroids at infinite shear rate are tabulated in tables 9a and 9b.* As the shear rate G is increased indefinitely, the conditions that $P \gg 1$ and $P \gg r_e^3 + r_e^{-3}$ will always be fulfilled eventually. Hence, the asymptotic results cited in tables 9a and 9b, which derive from the "weak" Brownian motion analysis of Section 9, show that the viscosity and normal stress functions each attain nonzero limiting values at infinite shear rate. Indeed, a slightly more accurate version of [9.10] shows, at least for the case where $r_p \gg 1$, that these limits for spheroids are

$$[\eta]_\infty = 0.315 \frac{r_p}{\ln 2r_p - 1.5}, \qquad [8.37a]$$

$$(\Sigma_1)_\infty = -\frac{r_p^2}{4(\ln 2r_p - 1.5)}, \qquad [8.37b]$$

$$(\Sigma_2)_\infty = \frac{r_p^4}{4(\ln 2r_p - 1.5)}. \qquad [8.37c]$$

Comparison of the asymptotic results of tables 9 with the "exact" values cited in tables 7 and 8 at $P = 60$ reveals that the infinite shear rate asymptotic limit is approached only very slowly as the Péclet number is increased, especially for the extreme aspect ratios $r_p \gg 1$ and $r_p \ll 1$, respectively, appropriate to prolate and oblate spheroids.

It is of interest to compare the intrinsic viscosity of a suspension composed of (touching) spherical doublets with that of the corresponding singlet suspension at the same solids volume fractional concentration. (In making the comparison it should be noted that when the spheres touch there exists no relative rotation or translation of the two spheres in a simple shear flow (O'Neill & Majumdar 1970), so that they behave like a single rigid body.) The intrinsic viscosity in the singlet case is, of course, given by Einstein's relation as $[\eta] = 2.5$. By contrast, from [8.5], tables 3 and 10, and [8.12], the intrinsic viscosities of the doublet suspension at zero and infinite shear rates are, respectively, $[\eta]_0 = 3.58$ and $[\eta]_\infty = 3.02$. The effect of including such particle-particle interactions is thereby to increase the resistance of the suspension to shear.

9. SIMPLE SHEAR FLOW. LARGE PÉCLET NUMBERS. $|B| < 1$.

By far, the majority of the theoretical rheological calculations reported in the literature pertain to the case where the rotary diffusion is dominant over the shear, i.e. $P = G/D_r \ll 1$

* On the other hand, when $P \gg 1$, but $r_e^3 \gg P \gg 1$ (or $r_e^{-3} \gg P \gg 1$), corresponding to the "intermediate" case discussed in Section 9, the appropriate values of the three viscometric functions may be obtained from [9.16] to [9.18] in conjunction with [9.15].

(Kirkwood 1967, Bird *et al.* 1971). This is the case discussed in Sections 7 and 8. Only recently have rigorous asymptotic techniques (Hinch & Leal 1972) been developed for the opposite case, where $P \gg 1$. Thus far, these methods have only been applied to the case of spheroids. However, with virtually no additional effort they may be applied to almost any axisymmetric particle, in particular those for which $|B| < 1$. Thus, the asymptotic results of Hinch & Leal (1972) are rendered applicable to such bodies by the simple expedient of replacing their particle axis ratio r_p by the equivalent axis ratio r_e.

The situation for which $|B| > 1$ (and $P \gg 1$) requires a different asymptotic analysis, and will be discussed in Section 10. All the bodies discussed in Section 3 possess the common feature that $|B| \le 1$, whence most of the important applications lie within the purview of the asymptotic analysis which follows.

The asymptotic solution of [8.19]–[8.21] for the case where

$$P \gg 1 \qquad [9.1]$$

subdivides naturally into two separate classes: (i) the "weak" Brownian motion case, characterized by

$$r_e^3 + r_e^{-3} \ll P, \qquad [9.2]$$

and (ii) the "intermediate" case, characterized by

$$r_e^3 + r_e^{-3} \gg P \gg 1. \qquad [9.3]$$

The "weak" Brownian motion case

Asymptotic results for this situation, correct to terms of $O(P^{-1})$, are easily abstracted from the results of Leal & Hinch, which pertain to the special case of spheroids. In particular, the three goniometric factors,* required for use in [8.5]–[8.7], are tabulated by Hinch & Leal (1972) and Leal & Hinch (1973) as a function of r_e (or B) and P, for the case of large P. These numerical values, being derived directly from the general equations [8.19]–[8.21] via definitions of the goniometric factors of the form [8.9] and the definition of r_e in [2.29], apply to any axisymmetric body for which $|B| < 1$. Hence, the applicability of Hinch & Leal's tabulation is not limited to spheroids.

Their tabulation is reproduced in table 10 for the range $1 \le r_e < \infty$ (i.e. $0 \le B \le 1$). Comparable results for the range $0 \le r_e \le 1$ (i.e. $-1 < B \le 0$) may be obtained from these by means of the set of transformations (cf. [8.25]–[8.26]),

$$r_e \to 1/r_e \quad (\text{i.e. } B \to -B), \qquad [9.4a]$$

$$\langle \sin^2 \theta \rangle \to \langle \sin^2 \theta \rangle, \qquad [9.4b]$$

$$\langle \sin^2 \theta \cos 2\phi \rangle \to -\langle \sin^2 \theta \cos 2\phi \rangle, \qquad [9.4c]$$

* In addition, Hinch & Leal (1972) tabulate values of $\langle \sin^4 \theta \sin 2\phi \rangle$, $\langle \sin^4 \theta \sin^2 2\phi \rangle$ and $\langle \sin^4 \theta \sin 4\phi \rangle$. However, in view of [8.10], these extra goniometrical factors are superfluous. The tabulated values of these quantities may, however, be utilized to examine the internal consistency of their tabulation. A few scattered checks of this nature revealed reasonable internal consistency.

Table 10. Numerical values of the goniometric factors to $O(P^{-1})$ for the "weak" Brownian motion case [9.1]–[9.2] as a function of the equivalent axis ratio in the range $1 \le r_e < \infty$ $(0 \le B \le 1)$.*

r_e	$\langle \sin^2\theta \rangle$	$\langle \sin^2\theta \cos 2\phi \rangle$	$P\langle \sin^2\theta \sin 2\phi \rangle$
1	0.667	0.0000	0.0000
1.05	0.66_3	−0.0194	0.1159
2	0.690	−0.2716	2.1653
3	0.727	−0.4186	4.9557
4	0.758	−0.5136	8.6551
5	0.784	−0.5810	13.302
7	0.823	−0.6718	25.487
10	0.862	−0.7530	51.090
16	0.905	−0.8366	128.92
25	0.936	−0.8918	312.55
50	0.968	−0.9388	1244.68
100	0.986	−0.9578	4994.81
∞	1.000	−1.0000	$\sim(1/2)r_e^2$

* Values of the goniometrical factors in the range $0 \le r_e \le 1$ $(-1 \le B \le 0)$ may be obtained from these by replacing r_e by $1/r_e$ and by changing the algebraic signs of $\langle \sin^2\theta \cos 2\phi \rangle$ and $P\langle \sin^2\theta \sin 2\phi \rangle$.

$$\langle \sin^2\theta \sin 2\phi \rangle \to -\langle \sin^2\theta \sin 2\phi \rangle. \qquad [9.4d]$$

Confirmation of the numerical accuracy of the goniometric factors presented in table 10 is provided by the numerical results cited in tables 6a, 6b and 6c. If attention is confined to the case where r_e lies in the range $\infty > r_e \ge 1$, then for $P = 60$ and 200, say, the inequalities [9.1] and [9.2] may be expected to apply to r_e values near unity, viz., $r_e = 1, 2, 3, \ldots$, the expected error becoming larger as one proceeds to the larger values of r_e. In table 11 we

Table 11. Comparison of the "exact" and approximate values of the goniometric factors for the "weak" Brownian motion case, [9.1] and[9.2].

	$\langle \sin^2\theta \rangle$			$-\langle \sin^2\theta \cos 2\phi \rangle$			$\langle \sin^2\theta \sin 2\phi \rangle$		
	"Exact" table 6c	Approx. table 7	Per cent error	"Exact" table 6b	Approx. table 7	Per cent error	"Exact" table 6a	Approx. table 7	Per cent error
r_e				$P = 60$					
1	0.667	0.667	0	0	0	0	0	0	0
2	0.692	0.690	0.3	0.266	0.272	2.3	0.0348	0.0361	3.7
3	0.725	0.727	0.3	0.396	0.419	5.5	0.0698	0.0826	18.3
4	0.748	0.758	1.3	0.465	0.514	10.6	0.0986	0.1443	46.4
				$P = 200$					
1	—	—	—	0	0	0	0	0	0
2	—	—	—	0.2716	0.2716	0.0	0.0108	0.01083	0.0
3	—	—	—	0.4154	0.4186	0.8	0.0235	0.0248	5.5
4	—	—	—	0.5021	0.5136	2.1	0.0358	0.0433	21.0

list the exact, Scheraga *et al.* goniometric factors for several values of r_e near unity, reproduced from tables 6a, 6b and 6c, as well as the (approximate) asymptotic Hinch & Leal factors, derived from table 10. Also listed are the percentage errors incurred by these approximate values.

Agreement between the approximate and exact values is quite good for those values of r_e nearest to unity. As anticipated, the discrepancy increases with increasing r_e. The reasonable agreement at the smaller values of r_e strongly supports the inherent accuracy of both the Scheraga *et al.* results and those of Hinch & Leal. Moreover, the quite good agreement at $P = 200$ for the smaller values of r_e suggests that the disclaimer by Scheraga *et al.* about the uncertain validity of their results at $P = 200$ is unduly cautious, at least at these small r_e values.

In addition to the values tabulated in table 10, Hinch & Leal (1972) and Leal & Hinch (1973) also derive the following asymptotic formulas, valid for $r_e \gg 1$ (i.e. $B \to 1$).*

$$\langle \sin^2 \theta \rangle = 1 - \frac{1.792}{r_e} + o\left(\frac{1}{r_e}\right), \qquad [9.5]$$

$$\langle \sin^2 \theta \cos 2\phi \rangle = -1 + \frac{3.0524}{r_e} + O\left(\frac{1}{r_e^2}\right), \qquad [9.6]$$

$$\langle \sin^2 \theta \sin 2\phi \rangle = \frac{1}{P}\left[\frac{r_e^2}{2} + O(1)\right] + O\left(\frac{r_e^5}{P^2}\right). \qquad [9.7]$$

From [9.2], these asymptotic results apply when

$$P^{1/3} \gg r_e \gg 1. \qquad [9.8]$$

These analytical relations agree well with the values tabulated in table 10 at the larger values of r_e.

The analogs of [9.5]–[9.7] for $r_e \ll 1$ (i.e. $B \to -1$) may be obtained from the above via the set of transformations [9.4]. These will apply when

$$P^{1/3} \gg r_e^{-1} \gg 1. \qquad [9.9]$$

In conjunction with [8.5]–[8.7], table 10 may be employed to calculate rheological properties for circumstances in which the inequality [9.2] applies. (See tables 9a and 9b for the case of spheroids.) Alternatively, asymptotic analytical expressions for these proper-

* Leal & Hinch (1973) also give the next term in the asymptotic expansion of [9.7] in inverse powers of P as

$$+\frac{r_e^5}{P^2}\left[0.47694 + O\left(\frac{1}{r_e}\right)\right] + O\left(\frac{r_e^{11}}{P^4}\right).$$

ties may be derived when the more stringent inequalities [9.8] or [9.9] hold. For example, in the case of long thin prolate spheroids (cf. [3.20] and [3.14]), we eventually find from [9.5] to [9.7] that*

$$[\eta] = 0.315\frac{r_p}{\ln r_p} + O\left(\frac{1}{P^2}\right), \qquad [9.10a]$$

$$[\tau_1] = -\frac{r_p^2}{4\ln r_p}\left(\frac{1}{P}\right) + O\left(\frac{1}{P^3}\right), \qquad [9.10b]$$

$$[\tau_2] = \frac{r_p^4}{4\ln r_p}\left(\frac{1}{P}\right) + O\left(\frac{1}{P^3}\right), \qquad [9.10c]$$

provided that

$$P^{1/3} \gg r_p \gg 1, \qquad [9.11]$$

where r_p is the particle axis ratio defined in [3.5]. The first and last of these expressions accord with the results of Hinch & Leal (1972), with account being taken of the exchange of the "1" and "2" indices from the notation of their paper (cf. footnote on page 242). However, in place of [9.10b] they obtain $[\tau_1] = o(r_p^4/\ln r_p)P^{-1}$. The discrepancy may perhaps stem from a possible failure on their part to take account of the asymptotic relation given in the footnote on the bottom of this page.

Note that [9.5]–[9.7] cannot be applied to either the "non-interacting" or "first-order" dumbbell (cf. [8.15] and [8.16]), since it has been assumed *a priori* that $r_e = \infty$ for such bodies (i.e. $B = 1$). Hence, it becomes impossible to find a Péclet number sufficiently large to satisfy the inequality [9.8]. (See, however, [9.22] and [9.13].) However, the numerical results tabulated in table 10 may still be employed to calculate the rheological properties of suspensions of more general dumbbells (cf. table 3), when the spheres comprising the dumbbell are sufficiently close to admit of strong hydrodynamic interactions. Such results apply only at Péclet numbers large enough to satisfy the inequality [9.2].

The "intermediate" case

For the intermediate case, corresponding to the inequality [9.3], Hinch & Leal (1972) have succeeded in obtaining an asymptotic solution valid only for the case where $r_e \gg 1$ (or, equivalently, by means of the transformation [9.4], for the case where $r_e \ll 1$, e.g. a circular disk, for which $r_e = r_p = 0$). From [9.3] this occurs when

$$r_e \gg P^{1/3} \gg 1. \qquad [9.12]$$

When this dual constraint is satisfied, the goniometric factors are of the forms

$$\langle \sin^2\theta \rangle = 1 - \frac{a}{P^{1/3}}, \qquad [9.13a]$$

* In the computation of $[\tau_1]$ it has been noted that the term $(B^{-1} - 1)$ appearing in [8.6] is asymptotically equal to $2/r_p^2$ (cf. [9.19]).

$$\langle \sin^2 \theta \cos 2\phi \rangle = -1 + \frac{b}{P^{1/3}}, \qquad [9.13b]$$

$$\langle \sin^2 \theta \sin 2\phi \rangle = \frac{c}{P^{1/3}}, \qquad [9.13c]$$

in which a, b, c are numerical constants, for which Hinch & Leal (1972) and Leal & Hinch (1973) give the approximate numerical values*

$$a \approx 2, \quad b \approx 4, \quad c \approx 0.4. \qquad [9.14]$$

These values are subject to considerable uncertainty. Leal & Hinch (1973) estimate the c coefficient to be no more reliable than ± 30 per cent. However, as is discussed in Appendix F, it is possible to employ the analysis of Stewart & Sørensen (1972) to estimate the following values for these coefficients:

$$a = 0.974, \quad b = 1.796, \quad c = 0.727. \qquad [9.15]$$

Yet another method, albeit approximate, for estimating these coefficients, due to Schwarz (1956), and discussed in Appendix F, yields

$$a = ?, \quad b \approx 1.22, \quad c \approx 0.71. \qquad [9.16]$$

Of the three different sets of coefficients tabulated, those given in [9.15] are likely to be most accurate since the Stewart & Sørensen numerics agree well with those of Scheraga *et al.* in their common domain of validity.

Some measure of the degree of accuracy of the asymptotic formulas [9.13], with coefficients given by [9.15], is furnished by comparison with the exact values of these goniometrical factors due to Scheraga *et al.* in tables 6a, 6b and 6c. It seems reasonable to assume that the dual inequality [9.12] is at least approximately satisfied by the values $r_e = 50$ and $P = 60$. For this value of P, [9.13] and [9.15] combine to yield

$$\langle \sin^2 \theta \cos 2\phi \rangle \approx -0.541, \quad \langle \sin^2 \theta \sin 2\phi \rangle \approx 0.186, \quad \langle \sin^2 \theta \rangle \approx 0.751.$$

These approximate values may be compared with the exact Scheraga *et al.* values of -0.582, 0.176 and 0.799, respectively, tabulated in tables 6a, 6b and 6c, or with the values -0.574, 0.172 and 0.797, respectively, tabulated in table 5 for $r_e = \infty$. Discrepancies here are of the order of 7 per cent, suggesting that a value of $P = 60$ is not sufficiently large for the asymptotic formulas [9.13] to apply with a high degree of accuracy. This is confirmed by the tabulation in table 5, which also reveals the quantitative inadequacies of the asymptotic formulas [9.13], even for P as large as 60.

* Actually, Hinch & Leal (1972) only directly give the values of a and c. The b value may be obtained indirectly by application of [8.10b] (with $B = 1$) in conjunction with the asymptotic value, $\langle \sin^4 \theta \sin^2 2\phi \rangle = 2P^{-1/3}$, given by Hinch & Leal.

Substitution of the goniometrical factors [9.13] into [8.5]–[8.7] yields

$$[\eta] = [\eta]_\infty + \frac{K}{P^{1/3}}, \qquad [9.17a]$$

$$[\tau_1] = \frac{K_1}{P^{1/3}}, \qquad [\tau_2] = \frac{K_2}{P^{1/3}}, \qquad [9.17b, c]$$

wherein, for circumstances in which the inequality [9.12] holds,

$$[\eta]_\infty = 5(Q_1 + Q_3^\circ) + (5/4)(B^{-1} - 1)(3Q_2 + 4Q_3^\circ), \qquad [9.18a]$$

$$K = (5/4)[3(a - b)Q_2 - 4bQ_3^\circ - b(B^{-1} - 1)(3Q_2 + 4Q_3^\circ)], \qquad [9.18b]$$

$$K_1 = (5/4)c[4Q_3^\circ + (B^{-1} - 1)(3Q_2 + 4Q_3^\circ)], \qquad [9.18c]$$

$$K_2 = -(5/4)c[6Q_2 + 4Q_3^\circ + (B^{-1} - 1)(3Q_2 + 4Q_3^\circ)], \qquad [9.18d]$$

are, in general, functions only of r_e.

Despite the fact that $B \to 1$ as $r_e \to \infty$, one must generally refrain from putting $B^{-1} - 1 = 0$ in these expressions. Rather, since

$$B^{-1} - 1 = \frac{2}{r_e^2 - 1} \sim \frac{2}{r_e^2} \quad \text{for } r_e \gg 1, \qquad [9.19]$$

the question of whether or not to put $B = 1$ in these equations depends critically upon the manner in which the material constants Q_1, Q_2 and Q_3° vary with r_e for large arguments r_e. For example, in the case of "non-interacting" dumbbells, it follows from [3.73] (being careful in passage to the limit $h = 0$), in conjunction with the values of a, b, c tabulated in [9.15], that*

$$[\eta] = 0.924 r_p^2 P^{-1/3}, \qquad [9.20a]$$

$$[\tau_1] = O(h r_p^2 P^{-4/3}), \qquad [9.20b]$$

$$[\tau_2] = 1.635 r_p^2 P^{-1/3}. \qquad [9.20c]$$

These results agree exactly with those of Stewart & Sørensen (1972), as may be verified by setting $h = 0$ in [8.15] and [F.1].

From [3.73g] and [3.71] we have that to terms of dominant order in the interaction parameter h,

$$r_e = \left(\frac{3}{2}\right)^{1/2}\left(\frac{3}{8}\right) h^{-1}. \qquad [9.21]$$

* It might appear from a comparison between [9.20a] and [9.17a] that $[\eta]_\infty = 0$. Actually this is not the case, since for the "non-interacting" dumbbell one finds from [9.18a] that

$$[\eta]_\infty = 5/2.$$

This term, however, being of $O(1)$ with respect to the parameter r_p, is negligible compared with the term of order r_p $(r_p/P^{1/3})$ appearing on the right-hand side of [9.20a]. This follows from the facts that $r_p/P^{1/3} \gg 1$ and $r_p \gg 1$ (cf. [9.12] and [3.73g]).

Hence, the inequality [9.12], required for the validity of [9.20], necessitates that

$$h^{-1} \gg P^{1/3} \gg 1. \qquad [9.22]$$

For any (large) specified value of P, this inequality can always be satisfied by choosing r_p sufficiently large, thereby making h sufficiently small.

In the case of a long thin prolate spheroid (cf. [3.20]), [9.18] becomes, asymptotically,

$$[\eta]_\infty = 2, \qquad K = (b-a)r_p^2/4 \ln r_p, \qquad K_1 = -c/\ln r_p, \qquad K_2 = cr_p^2/2 \ln r_p.$$

Insertion of these into [9.17], with use of Leal and Hinch's values for the constants a, b, c, given in [9.14],* yields

$$[\eta] = (0.5r_p^2/\ln r_p)P^{-1/3}, \qquad [9.23a]$$

$$[\tau_1] = -(0.4/\ln r_p)P^{-1/3}, \qquad [9.23b]$$

$$[\tau_2] = (0.2r_p^2/\ln r_p)P^{-1/3}. \qquad [9.23c]$$

The value $[\eta]_\infty = 2$, being of $O(1)$ with respect to the large parameter r_p, has been neglected in obtaining [9.23a]. In view of the inequality [9.12] (with $r_e = r_p$ for the spheroid), such a term of order unity is negligible compared with the term of order $(r_p/\ln r_p)(r_p/P^{1/3})$ appearing explicitly on the right-hand side of [9.23a].

Equations [9.23] agree with the original spheroid results of Hinch & Leal (1972) for the "intermediate" case (cf. footnote on page 242 for minor notational differences), except that they give $[\tau_1] = o(1)(r_p^2/\ln r_p)P^{-1/3}$ in place of [9.23b]. Their error here is likely due to the same source as that discussed in connection with [9.10b], namely their probable use of the relation $B^{-1} - 1 = 0$, rather than the correct relation indicated in [9.19] for $r_e \gg 1$. In both cases their expressions for $[\tau_1]$ are too large by a factor of r_p^2.

10. SIMPLE SHEAR FLOW. LARGE PÉCLET NUMBERS. $|B| > 1$.

From [8.21] the equations governing rotation of an axisymmetric body suspended in the simple shear flow [8.1] are

$$\dot{\theta} = \tfrac{1}{4}BG \sin 2\theta \sin 2\phi, \qquad [10.1]$$

and

$$\dot{\phi} = \tfrac{1}{2}G(1 + B\cos 2\phi). \qquad [10.2]$$

In addition, as readily follows from [3.3], the body rotates about its symmetry axis with an angular velocity

$$\Omega_e = \tfrac{1}{2}G\cos\theta, \qquad [10.3]$$

where $\mathbf{\Omega}_e = \mathbf{\Omega} \cdot \mathbf{e}$.

* The a, b and c values given in [9.15] are presumably more accurate than those of Hinch & Leal. We have merely used the latter's values for the purpose of comparing [9.23] with the original calculations of Hinch & Leal (1972).

For $|B| < 1$, [10.1] and [10.2] reveal that in the absence of rotary diffusion the body undergoes a time-periodic rotation of the type first described by Jeffery (1922). In contrast, when $|B| > 1$ the body undergoes an aperiodic motion, ultimately adopting a stable terminal orientation $(\theta^\infty, \phi^\infty)$, characterized by

$$\dot{\theta}^\infty = 0, \quad \dot{\phi}^\infty = 0. \qquad [10.4]$$

This terminal orientation is easily calculated by putting [10.1] and [10.2] equal to zero, and determining, by means of a linear stability analysis (Brenner 1972c) which roots $(\theta^\infty, \phi^\infty)$ of the resulting equations are stable to small perturbations in orientation. Alternatively, one can integrate [10.1] and [10.2] and pass to the limit as $t \to \infty$ (Bretherton 1962, Brenner 1972c).

For $B \geq 1$ the stable orientation is found to be

$$\theta^\infty = \pi/2, \quad \tan\phi^\infty = r_e, \qquad [10.5a, b]$$

where r_e is the quantity defined by [2.31]. As B varies from 1 to ∞, r_e varies from ∞ to 1. Thus, in its stable terminal state, the particle lies in the $x_1 - x_2$ plane, and there its symmetry axis makes a positive angle ϕ^∞ with the x_1 axis, lying either in the range

$$\pi/2 \geq \phi^\infty \geq \pi/4, \qquad [10.6a]$$

or

$$\pi/2 \geq \phi^\infty - \pi/2 \geq \pi/4, \qquad [10.6b]$$

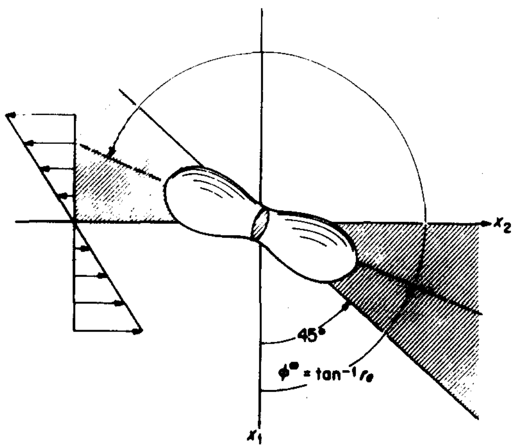

Figure 13. Terminal orientation of an axisymmetric body characterized by $B > 1$ when suspended in a simple shear flow.

as in figure 13. Depending upon the specified r_e value, these stable orientations lie somewhere within the shaded regions shown in figure 13. These two different stable orientations merely correspond to the two opposite ends of the symmetry axis of the body. When one end points in the direction given by [10.6a] the other end points in the opposite direction, given by [10.6b]. From [5.5] these orientations correspond to

$$\mathbf{e}^{\infty} = \mathbf{i}_1 \cos \phi^{\infty} + \mathbf{i}_2 \sin \phi^{\infty}, \qquad [10.7]$$

in which either

$$\cos \phi^{\infty} = \frac{1}{(r_e^2 + 1)^{1/2}}, \quad \sin \phi^{\infty} = \frac{r_e}{(r_e^2 + 1)^{1/2}}, \qquad [10.8a]$$

or

$$\cos \phi^{\infty} = -\frac{1}{(r_e^2 + 1)^{1/2}}, \quad \sin \phi^{\infty} = -\frac{r_e}{(r_e^2 + 1)^{1/2}}. \qquad [10.8b]$$

The former corresponds to that end lying in the first quadrant, and the latter to the opposite end, lying in the third quadrant.

For $B \le -1$ the stable terminal orientation is (Brenner 1972c)

$$\theta^{\infty} = \pi/2, \quad \tan \phi^{\infty} = -r_e, \qquad [10.9a, b]$$

with r_e defined as in [2.31]. As B varies from -1 to $-\infty$, r_e varies from 0 to 1. Again, the particle lies in the $x_1 - x_2$ plane, as in figure 14, and there makes a positive angle ϕ^{∞} with the x_1 axis, lying either in the range

$$3\pi/4 \ge \phi^{\infty} \ge \pi/2, \qquad [10.10a]$$

or

$$3\pi/4 \ge \phi^{\infty} - \pi/2 \ge \pi/2. \qquad [10.10b]$$

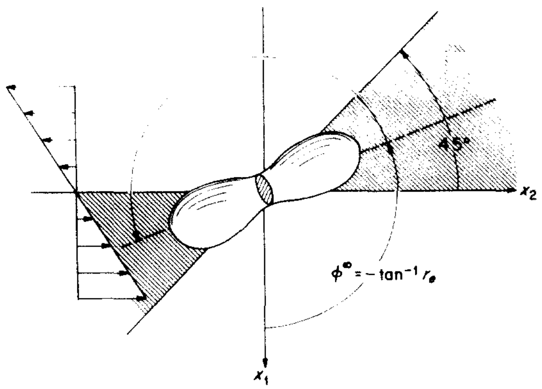

Figure 14. Terminal orientation of an axisymmetric body characterized by $B < -1$ when suspended in a simple shear flow.

In [10.7] this stable orientation corresponds either to

$$\cos\phi^{\infty} = -\frac{1}{(r_e^2+1)^{1/2}}, \quad \sin\phi^{\infty} = \frac{r_e}{(r_e^2+1)^{1/2}}, \qquad [10.11a]$$

or

$$\cos\phi^{\infty} = \frac{1}{(r_e^2+1)^{1/2}}, \quad \sin\phi^{\infty} = -\frac{r_e}{(r_e^2+1)^{1/2}}. \qquad [10.11b]$$

Since the suspended particles in a dilute suspension for which $|B| \geq 1$ adopt preferred terminal orientations in the absence of rotary Brownian motion, the orientational distribution function for this case is the Dirac delta function distribution

$$f(\mathbf{e}) = (1/2)[\delta(\mathbf{e}+\mathbf{e}^{\infty}) + \delta(\mathbf{e}-\mathbf{e}^{\infty})], \qquad [10.12]$$

there being no distinction between the directions $\pm\mathbf{e}^{\infty}$. Consequently, in the limit where $P = G/D_r \to \infty$,

$$\langle\mathbf{ee}\rangle = \mathbf{e}^{\infty}\mathbf{e}^{\infty}, \qquad [10.13]$$

and thus

$$\langle\sin^2\theta\rangle = \sin^2\theta^{\infty} \equiv 1, \qquad [10.14a]$$

$$\langle\sin^2\theta\sin 2\phi\rangle = \sin^2\theta^{\infty}\sin 2\phi^{\infty} = B^{-1}(B^2-1)^{1/2}, \qquad [10.14b]$$

$$\langle\sin^2\theta\cos 2\phi\rangle = \sin^2\theta^{\infty}\cos 2\phi^{\infty} = -B^{-1}, \qquad [10.14c]$$

valid for both $B \geq 1$ and $B \leq -1$ (cf. [9.4]). These goniometric factors may be employed in [8.5] to [8.7] to calculate rheological properties in simple shear for the case where $|B| \geq 1$, provided that rotary Brownian motion is negligible.

Effects of weak Brownian motion

To incorporate the effects of small rotary Brownian movement into the analysis we follow the general methods of Hinch (1971). In the absence of Brownian motion the general mechanical equations of motion of an axisymmetric particle are (Brenner & Condiff 1974)

$$\dot{\mathbf{e}} = (\mathbf{I}-\mathbf{ee})\cdot\mathbf{H}\cdot\mathbf{e}, \qquad [10.15]$$

in which $\mathbf{H}$ is the dyadic

$$\mathbf{H} = \mathbf{\Lambda} + B\mathbf{S}, \qquad [10.16]$$

and $\dot{\mathbf{e}} \equiv d\mathbf{e}/dt$ is the time rate of change of the orientation of the symmetry axis of the particle as measured by an observer fixed in space.

In the particular case of the simple shear flow [8.1], equation [10.15] is equivalent to [10.1] and [10.2]. However, in the interests of generality, we shall refrain for the time being from introducing particular values of $\mathbf{\Lambda}$ and $\mathbf{S}$.

When Brownian motion is absent the stationary orientations of the particle $\mathbf{e}^{\infty}$ (and $-\mathbf{e}^{\infty}$) are defined by the condition that

$$\dot{\mathbf{e}}^{\infty} = 0. \quad [10.17]$$

Hence, from [10.15], $\mathbf{e}^{\infty}$ may be obtained from the relation

$$(\mathbf{I} - \mathbf{e}^{\infty}\mathbf{e}^{\infty}) \cdot \mathbf{H} \cdot \mathbf{e}^{\infty} = 0. \quad [10.18]$$

Since $\mathbf{I} - \mathbf{e}^{\infty}\mathbf{e}^{\infty}$ is an annihilator for all vectors parallel to $\mathbf{e}^{\infty}$, it follows that the general solution of [10.18] corresponds to the requirement that the vector $\mathbf{H} \cdot \mathbf{e}^{\infty}$ lie parallel to $\mathbf{e}^{\infty}$, i.e.

$$\mathbf{H} \cdot \mathbf{e}^{\infty} = h\mathbf{e}^{\infty}. \quad [10.19]$$

Equivalently,

$$(\mathbf{H} - \mathbf{I}h) \cdot \mathbf{e}^{\infty} = 0. \quad [10.20]$$

The scalar h is therefore an eigenvalue of $\mathbf{H}$, and $\mathbf{e}^{\infty}$ is the corresponding eigenvector. For [10.20] to possess a nontrivial solution, h must be a solution of the characteristic equation

$$\det(\mathbf{H} - \mathbf{I}h) = 0. \quad [10.21]$$

For specified values of $\mathbf{\Lambda}$, $\mathbf{S}$ and B this represents, in general, a cubic equation in h, possessing three roots. These roots may either all be real, or else one root may be real with the other two complex conjugates.

In the case of the simple shear flow [8.1],

$$\mathbf{H} = \tfrac{1}{2}BG(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1) + \tfrac{1}{2}G(\mathbf{i}_2\mathbf{i}_1 - \mathbf{i}_1\mathbf{i}_2). \quad [10.22]$$

The three eigenvectors are readily found to be

$$h_1 = H, \quad h_2 = -H, \quad h_3 = 0, \quad [10.23]$$

in which

$$H = \tfrac{1}{2}G(B^2 - 1)^{1/2} \quad [10.24]$$

Stationary states are possible only when H is real. This occurs only when $|B| > 1$. When $|B| < 1$, H is purely complex, and no stationary states are possible, at least for the case of simple shear flow.

Following [5.5] the eigenvectors $\mathbf{e}_1^{\infty}$ and $\mathbf{e}_2^{\infty}$ appropriate to h_1 and h_2, respectively, may be written generally as

$$\mathbf{e}_j^{\infty} = (\mathbf{i}_1 \cos\phi_j^{\infty} + \mathbf{i}_2 \sin\phi_j^{\infty}) \sin\theta_j^{\infty} + \mathbf{i}_3 \cos\theta_j^{\infty} \qquad (j = 1, 2), \quad [10.25]$$

The eigenvector $\mathbf{e}_3^{\infty}$ is indeterminate and irrelevant in view of the two-dimensional nature of $\mathbf{H}$ for a simple shear flow. Introduction of [10.22]–[10.25] into [10.19] shows that

$$\theta_1^{\infty} = \theta_2^{\infty} = \pi/2 = \theta^{\infty}, \text{ say}, \quad [10.26]$$

and

$$\tan \phi_1^\infty = r_e \operatorname{sgn} B, \qquad [10.27]$$

$$\tan \phi_2^\infty = -r_e \operatorname{sgn} B, \qquad [10.28]$$

whence [10.25] may be written as

$$\mathbf{e}_j^\infty = \mathbf{i}_1 \cos \phi_j^\infty + \mathbf{i}_2 \sin \phi_j^\infty. \qquad [10.29]$$

To ascertain which, if either, of the two normalized eigenvectors $\mathbf{e}_1^\infty$ and $\mathbf{e}_2^\infty$ represents a stable stationary state we perform a linearized stability analysis. Write

$$\mathbf{e} = \mathbf{e}^\infty + \boldsymbol{\Delta}, \qquad [10.30]$$

in which $\Delta \ll 1$, where $\Delta = |\boldsymbol{\Delta}|$. Here, the vector $\boldsymbol{\Delta}$ represents a small perturbation about the stationary state $\mathbf{e}^\infty$. For convenience we have temporarily dropped the subscript j on $\mathbf{e}^\infty$ and $\boldsymbol{\Delta}$. Set

$$\boldsymbol{\Delta} = \boldsymbol{\Delta}_\parallel + \boldsymbol{\Delta}_\perp,$$

wherein $\boldsymbol{\Delta}_\parallel$ and $\boldsymbol{\Delta}_\perp$ lie parallel and perpendicular, respectively, to $\mathbf{e}^\infty$. Since $\mathbf{e} \cdot \mathbf{e} = \mathbf{e}^\infty \cdot \mathbf{e}^\infty = 1$, we find upon dot multiplication of [10.30] by itself that $\mathbf{e}^\infty \cdot \boldsymbol{\Delta} = O(\Delta^2)$. Alternatively, since $\mathbf{e}^\infty \cdot \boldsymbol{\Delta}_\perp = 0$, then $\mathbf{e}^\infty \cdot \boldsymbol{\Delta}_\parallel = O(\Delta_\parallel^2 + \Delta_\perp^2)$, where $\Delta_\parallel$ and $\Delta_\perp$ are the magnitudes of the corresponding vectors. Inasmuch as $\mathbf{e}^\infty$ and $\boldsymbol{\Delta}_\parallel$ are colinear and $|\mathbf{e}^\infty| = 1$, then $\mathbf{e}^\infty \cdot \boldsymbol{\Delta}_\parallel = \Delta_\parallel$. Thereby we obtain $\Delta_\parallel = O(\Delta_\perp^2)$. In consequence of this, only perturbations lying in the plane perpendicular to $\mathbf{e}^\infty$ need be considered in the linearized perturbation analysis. Thus, [10.30] may be replaced by

$$\mathbf{e} = \mathbf{e}^\infty + \boldsymbol{\Delta}_\perp + O(\Delta^2), \qquad [10.31]$$

wherein

$$\mathbf{e}^\infty \cdot \boldsymbol{\Delta}_\perp = 0. \qquad [10.32]$$

Substitution of [10.31] into [10.15], with use of [10.17]–[10.20] and [10.32], ultimately yields

$$\dot{\boldsymbol{\Delta}}_\perp = \mathbf{A} \cdot \boldsymbol{\Delta}_\perp + O(\Delta^2), \qquad [10.33]$$

where $\mathbf{A}$ is the dyadic*

$$\mathbf{A} = (\mathbf{I} - \mathbf{e}^\infty \mathbf{e}^\infty) \cdot \mathbf{H} - (\mathbf{I} - \mathbf{e}^\infty \mathbf{e}^\infty) h. \qquad [10.34]$$

* In view of [10.32], equation [10.33] could also be written as

$$\dot{\boldsymbol{\Delta}}_\perp = \mathbf{A}' \cdot \boldsymbol{\Delta}_\perp + O(\Delta^2),$$

in which

$$\mathbf{A}' = (\mathbf{I} - \mathbf{e}^\infty \mathbf{e}^\infty) \cdot \mathbf{H} - \mathbf{I} h$$

is a complete (i.e. three-dimensional) dyadic.

It readily follows from [10.18] along with $\mathbf{e}^{\infty} \cdot \mathbf{e}^{\infty} = 1$ that

$$\mathbf{A} \cdot \mathbf{e}^{\infty} = 0 \quad \text{and} \quad \mathbf{e}^{\infty} \cdot \mathbf{A} = 0, \qquad [10.35]$$

so that $\mathbf{A}$ is a planar (i.e. two-dimensional) dyadic whose components lie entirely in the plane perpendicular to $\mathbf{e}^{\infty}$.

As a solution of the linearized equation [10.33] we try

$$\mathbf{\Delta}_{\perp} = c\mathbf{a}e^{At}, \qquad [10.36]$$

where c is a constant scalar and $\mathbf{a}$ is a constant unit vector lying in the plane perpendicular to $\mathbf{e}^{\infty}$. Substitution into [10.33] yields

$$\mathbf{A} \cdot \mathbf{a} = A\mathbf{a}, \qquad [10.37]$$

i.e.

$$(\mathbf{A} - \mathbf{I}_{\perp}A) \cdot \mathbf{a} = 0, \qquad [10.38]$$

where $\mathbf{I}_{\perp} \equiv \mathbf{I} - \mathbf{e}^{\infty}\mathbf{e}^{\infty}$ is the two-dimensional idemfactor for vectors lying in a plane perpendicular to $\mathbf{e}^{\infty}$. For [10.38] to possess a nontrivial solution it is required that

$$\det(\mathbf{A} - \mathbf{I}_{\perp}A) = 0, \qquad [10.39]$$

whence the roots A_1 and A_2 of this quadratic equation are the two eigenvalues of the planar dyadic $\mathbf{A}$, and the (unit) vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are its eigenvectors. The eigenvalues are either both real (as is always true when $\mathbf{A}$ is symmetric) or else they are complex conjugates. By linear superposition, the general solution of [10.33] may be written as

$$\mathbf{\Delta}_{\perp} = c_1\mathbf{a}_1e^{A_1t} + c_2\mathbf{a}_2e^{A_2t}, \qquad [10.40]$$

where the constants c_1 and c_2 are determined by the initial conditions. Stability of the stationary state requires that the real parts of A_1 and A_2 be negative. Necessary and sufficient conditions for this to be so are

$$\operatorname{tr}\mathbf{A} < 0 \quad \text{and} \quad \det\mathbf{A} > 0. \qquad [10.41]$$

In the special case where $\mathbf{A}$ is symmetric, stability simply requires that

$$A_1 < 0 \quad \text{and} \quad A_2 < 0. \qquad [10.42]$$

In order to obtain an explicit formula for $\mathbf{A}$ for the simple shear flow case, it is convenient to introduce a right-handed system of mutually perpendicular unit vectors $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$:

$$\mathbf{a}_1 = \mathbf{i}_3, \quad \mathbf{a}_2 = \mathbf{e}^{\infty} \times \mathbf{i}_3, \quad \mathbf{a}_3 = \mathbf{e}^{\infty}. \qquad [10.43]$$

Here, $\mathbf{i}_3$ is the unit vector parallel to the vorticity vector of the simple shear flow [8.1], and $\mathbf{e}^{\infty}$ is given by [10.29] for this flow. The unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ lie in the plane perpendicular to $\mathbf{e}^{\infty}$. With use of [10.29] it follows from [10.43] that

$$\mathbf{a}_2 = \mathbf{i}_1 \sin\phi_j^{\infty} - \mathbf{i}_2 \cos\phi_j^{\infty}, \qquad [10.44a]$$

$$\mathbf{a}_3 = \mathbf{i}_1 \cos\phi_j^{\infty} + \mathbf{i}_2 \sin\phi_j^{\infty}, \qquad [10.44b]$$

or, solving these for $\mathbf{i}_1$ and $\mathbf{i}_2$,

$$\mathbf{i}_1 = \mathbf{a}_2 \sin \phi_j^\infty + \mathbf{a}_3 \cos \phi_j^\infty, \qquad [10.45a]$$

$$\mathbf{i}_2 = -\mathbf{a}_2 \cos \phi_j^\infty + \mathbf{a}_3 \sin \phi_j^\infty. \qquad [10.45b]$$

In the present system,

$$\mathbf{I} - \mathbf{e}_j^\infty \mathbf{e}_j^\infty = \mathbf{a}_1\mathbf{a}_1 + \mathbf{a}_2\mathbf{a}_2. \qquad [10.46]$$

From [10.22] to [10.24], [10.45], [10.46] and [10.34] follows

$$\mathbf{A}_1 = -H\mathbf{a}_1\mathbf{a}_1 - 2H\mathbf{a}_2\mathbf{a}_2, \qquad [10.47a]$$

and

$$\mathbf{A}_2 = H\mathbf{a}_1\mathbf{a}_1 + 2H\mathbf{a}_2\mathbf{a}_2, \qquad [10.47b]$$

in which we have used the relations

$$\tan^2 \phi_j^\infty = \frac{B+1}{B-1} \quad (j = 1, 2),$$

$$\cos 2\phi_j^\infty = -\frac{1}{B} \quad (j = 1, 2),$$

and

$$B \sin 2\phi_1^\infty = (B^2 - 1)^{1/2}, \quad B \sin 2\phi_2^\infty = -(B^2 - 1)^{1/2},$$

derived from [10.27] and [10.28]. Since $\mathbf{A}$ is symmetric in both cases, it may be written in terms of its normalized eigenvectors and eigenvalues as

$$\mathbf{A} = \mathbf{a}_1\mathbf{a}_1 A_1 + \mathbf{a}_2\mathbf{a}_2 A_2. \qquad [10.48]$$

Comparison with [10.47] shows that the stability criterion [10.42] is satisfied only by $\mathbf{A}_1$. Hence, we conclude that only $\mathbf{e}_1^\infty$ is stable, whence it follows from [10.26] and [10.27] that the stable orientation is

$$\theta^\infty = \pi/2, \quad \tan \phi^\infty = r_e \operatorname{sgn} B. \qquad [10.49]$$

This result accords with [10.5] for $B \geq 1$ and [10.9] for $B \leq -1$.

In summary, the stable terminal orientation $\mathbf{e}^\infty$ for the simple shear flow [8.1] is given by [10.7] where ϕ^∞ is the value defined by [10.49] (or by [10.5b] for $B \geq 1$ and [10.9b] for $B \leq -1$). For this stable state the $\mathbf{A}$ value is given by [10.48], in which

$$A_1 = -H, \quad A_2 = -2H. \qquad [10.50]$$

When rotary Brownian motion is sensible the orientational distribution function satisfies the differential equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{e}} \cdot (\dot{\mathbf{e}} f) = D_r \frac{\partial^2 f}{\partial \mathbf{e} \cdot \partial \mathbf{e}}, \qquad [10.51]$$

subject to the normalization condition

$$\oint f \, d^2\mathbf{e} = 1, \qquad [10.52]$$

with $\dot{\mathbf{e}}$ given generally by [10.15]. For $D_r = 0$ the solution of this differential equation is the delta function distribution [10.12], in which $\mathbf{e}^{\infty}$ is the stable terminal orientation.

When the Brownian motion is weak, but nonzero, it may be regarded as a small perturbation about the terminal state $\mathbf{e}^{\infty}$. Hence, from [10.31] and [10.17] we have that

$$\dot{\mathbf{e}} = \dot{\mathbf{\Delta}}_{\perp} = \mathbf{A} \cdot \mathbf{\Delta}_{\perp} \qquad [10.53]$$

to terms of lowest order in the small perturbation $\mathbf{\Delta}$. Furthermore, from [10.31], $\partial \mathbf{e} = \partial \mathbf{\Delta}_{\perp}$ since $\mathbf{e}^{\infty}$ is a constant. At steady state the distribution function for the case of weak Brownian motion thereby satisfies the differential equation

$$\frac{\partial}{\partial \mathbf{\Delta}_{\perp}} \cdot (\mathbf{A} \cdot \mathbf{\Delta}_{\perp} f) = D_r \frac{\partial^2 f}{\partial \mathbf{\Delta}_{\perp} \cdot \partial \mathbf{\Delta}_{\perp}} \qquad [10.54]$$

locally. This equation can be written as

$$\frac{\partial}{\partial \mathbf{\Delta}_{\perp}} \cdot \mathbf{j}_{\Delta} = 0, \qquad [10.55]$$

in which

$$\mathbf{j}_{\Delta} = -D_r \frac{\partial f}{\partial \mathbf{\Delta}_{\perp}} + \mathbf{A} \cdot \mathbf{\Delta}_{\perp} f \qquad [10.56]$$

is the rotary perturbation flux (Brenner & Condiff 1974).

When $\mathbf{A}$ is symmetric, such as is true for the case of a simple shear flow, these relations admit of the solution

$$\mathbf{j}_{\Delta} = 0, \qquad [10.57]$$

corresponding to a balance between the rotary diffusive and convective fluxes (Brenner & Condiff 1974). Integration of [10.56] then yields the multivariate Gaussian distribution

$$f = K^{-1} \exp(-\mathbf{\Delta}_{\perp} \cdot \mathbf{C} \cdot \mathbf{\Delta}_{\perp}), \qquad [10.58]$$

in which $\mathbf{C}$ is the symmetric dyadic

$$\mathbf{C} = -\mathbf{A}/2D_r, \qquad [10.59]$$

and $K \equiv K(\mathbf{C})$ is the normalization constant

$$K \equiv \oint \exp(-\mathbf{\Delta}_{\perp} \cdot \mathbf{C} \cdot \mathbf{\Delta}_{\perp}) \, d^2\mathbf{e}, \qquad [10.60]$$

deriving from [10.52].* This constant is evaluated in Appendix H, with the result that

$$K = 2\pi(\det \mathbf{C})^{-1/2}. \qquad [10.61]$$

Independently of whether or not $\mathbf{A}$ is symmetric, $\mathbf{C}$ is a symmetric planar dyadic* lying in the plane normal to $\mathbf{e}^\infty$. It therefore satisfies

$$\mathbf{e}^\infty \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{e}^\infty = 0.$$

In consequence of this, and the fact that $\mathbf{\Delta}_\perp = \mathbf{e} - \mathbf{e}^\infty$, [10.58] may be written in the invariant form

$$f(\mathbf{e}) = \frac{1}{2\pi}(\det \mathbf{C})^{1/2} \exp(-\mathbf{e} \cdot \mathbf{C} \cdot \mathbf{e}). \qquad [10.62]$$

In the simple shear flow case it follows from [10.48], [10.59], [10.50] and [10.24] that

$$\mathbf{C} = \mathbf{a}_1\mathbf{a}_1 C_1 + \mathbf{a}_2\mathbf{a}_2 C_2, \qquad [10.63]$$

in which

$$C_1 = \tfrac{1}{4}\Lambda, \quad C_2 = \tfrac{1}{2}\Lambda, \qquad [10.64]$$

where

$$\Lambda = P(B^2 - 1)^{1/2} \gg 1, \qquad [10.65]$$

with

$$P = G/D_r \qquad [10.66]$$

the rotary Péclet number. According to the unsteady solution of [10.1] and [10.2] for $|B| > 1$ (Bretherton 1962, Brenner 1972c), the hydrodynamic relaxation time τ_H for approach to the terminal orientation is (cf. [10.40], [10.50] and [10.24])

$$\tau_H = \frac{2}{G(B^2 - 1)^{1/2}}. \qquad [10.67]$$

On the other hand, the diffusional relaxation time τ_D is

$$\tau_D = 6/D_r. \qquad [10.68]$$

* In the more general case where $\mathbf{A}$ is not symmetrical the solution is still of the form [10.58] (with $\mathbf{C}$ symmetrical), but $\mathbf{C}$ is no longer given by [10.59]. This planar dyadic may be determined by substitution of [10.58] into [10.54], yielding

$$\mathbf{v} \cdot (\mathbf{C}^{-1} \cdot \mathbf{A}^\dagger + \mathbf{A} \cdot \mathbf{C}^{-1} + 4D_r\mathbf{I}_\perp) \cdot \mathbf{v} = \mathrm{tr}(\mathbf{A} + 2D_r\mathbf{C}),$$

in which $\mathbf{v} = \mathbf{C} \cdot \mathbf{\Delta}_\perp$ and $\mathbf{I}_\perp$ is the two-dimensional idemfactor, $\mathbf{I} - \mathbf{e}^\infty\mathbf{e}^\infty$. In order that this relation may apply for an arbitrary value of $\mathbf{\Delta}_\perp$ we require that $\mathbf{C}$ satisfy

$$\mathbf{C}^{-1} \cdot \mathbf{A}^\dagger + \mathbf{A} \cdot \mathbf{C}^{-1} + 4D_r\mathbf{I}_\perp = 0, \qquad \mathrm{tr}(\mathbf{A} + 2D_r\mathbf{C}) = 0,$$

in agreement with Hinch (1971). The latter condition is automatically implied by the former. The normalization constant in [10.60] applies whether or not $\mathbf{A}$ is symmetric, though it is incorrectly stated in Hinch's (1971) thesis with $(\det \mathbf{C})^{-1}$ appearing in place of $(\det \mathbf{C})^{-1/2}$.

Consequently, the parameter

$$\Lambda = \frac{1}{3}\frac{\tau_D}{\tau_H} \qquad [10.69]$$

is the ratio of relaxation times for the rotary diffusion and convection.

With use of [5.5] and [10.44] (upon deleting the subscript j from the latter), equation [10.62] adopts the form

$$f(\theta, \phi) = \frac{\Lambda}{4\pi\sqrt{2}} \exp\left[-\tfrac{1}{4}\Lambda\{\cos^2\theta + 2\sin^2\theta\sin^2(\phi - \phi^\infty)\}\right], \qquad [10.70]$$

where, from [10.49],

$$\phi^\infty = (\operatorname{sgn} B)\tan^{-1}\left(\frac{B+1}{B-1}\right)^{1/2}. \qquad [10.71]$$

In the limit where $\Lambda \to \infty$, this distribution goes to zero for all (θ, ϕ) except at the critical orientation, $\theta = \theta^\infty = \pi/2$, $\phi = \phi^\infty$. This is consistent with the Dirac delta function character of the distribution [10.12] in the limit where the Brownian motion is negligible.

A general expression for the second orientational moment $\langle \mathbf{ee} \rangle$ required in the rheological calculations is derived in Appendix H, the result being

$$\langle \mathbf{ee} \rangle = [1 - \tfrac{1}{2}\operatorname{tr}(\mathbf{C}^{-1})]\mathbf{e}^\infty\mathbf{e}^\infty + \tfrac{1}{2}\mathbf{C}^{-1}, \qquad [10.72]$$

valid in the limit of small rotary Brownian movement.* For the simple shear flow [8.1] this gives

$$\langle \mathbf{ee} \rangle = (1 - 3\Lambda^{-1})\mathbf{e}^\infty\mathbf{e}^\infty + 2\Lambda^{-1}(2\mathbf{a}_1\mathbf{a}_1 + \mathbf{a}_2\mathbf{a}_2). \qquad [10.73]$$

This relation gives the first-order correction to [10.13] arising from the Brownian rotation. It can be written out in the $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ system appropriate to the simple shear flow [8.1] by use of [10.7], [10.44] (upon deleting the affix j), and [10.49]. In this manner it follows that

$$\langle \sin^2\theta \rangle = (\mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2):\langle \mathbf{ee} \rangle = 1 - 2\Lambda^{-1}, \qquad [10.74a]$$

$$\langle \sin^2\theta \sin 2\phi \rangle = (\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1):\langle \mathbf{ee} \rangle = B^{-1}(B^2 - 1)^{1/2}(1 - 4\Lambda^{-1}), \qquad [10.74b]$$

and

$$\langle \sin^2\theta \cos 2\phi \rangle = (\mathbf{i}_1\mathbf{i}_1 - \mathbf{i}_2\mathbf{i}_2):\langle \mathbf{ee} \rangle = -B^{-1}(1 - 4\Lambda^{-1}). \qquad [10.74c]$$

These represent the $O(\Lambda^{-1})$ corrections to [10.14]. They apply for both $B > 1$ and $B < -1$. Equations [10.74] may be utilized in [8.5]–[8.7] to determine rheological properties for $|B| > 1$ to terms of the first order in D_r.

* Equation [10.72] holds in general, even if $\mathbf{A}$ is not symmetric, since $\mathbf{a}_1$, $\mathbf{a}_2$, C_1 and C_2 may also be interpreted as being the principal axes and principal values, respectively, of the symmetric planar dyadic $\mathbf{C}$.

11. GENERAL TWO-DIMENSIONAL HOMOGENEOUS SHEARING FLOWS

Sections 8–10 have furnished a detailed analysis of the rheological properties of dilute suspensions of axisymmetric Brownian particles undergoing simple shear. We will demonstrate in this section, by a simple reinterpretation of various parameters, that these simple shear results can be applied to general, two-dimensional, homogeneous shearing flows. In particular it will be shown *inter alia* that the results of Scheraga *et al.* (1951, 1955) giving numerical values for the orientational distribution function, and the various moments thereof, may be employed for any two-dimensional flow, by reassigning a different interpretation to the parameters B (or r_e) and G upon which the distribution function f depends. This same reinterpretation enables us to carry over the asymptotic, large Péclet number analyses of Sections 9 and 10 to these more general flows.

Let (x'_1, x'_2, x'_3) represent an arbitrary system of rectangular Cartesian coordinates fixed in space, and consider the general incompressible two-dimensional flow

$$\mathbf{u} = \mathbf{i}'_1 u'_1(x'_1, x'_2) + \mathbf{i}'_2 u'_2(x'_1, x'_2), \tag{11.1}$$

$$\frac{\partial u'_1}{\partial x'_1} + \frac{\partial u'_2}{\partial x'_2} = 0, \tag{11.2}$$

taking place in the $x'_1 - x'_2$ plane. Here, $(\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3)$ are a right-handed triad of mutually perpendicular unit vectors. Since

$$\boldsymbol{\omega} = \tfrac{1}{2}\nabla \times \mathbf{u}, \tag{11.3}$$

then for this flow,

$$\boldsymbol{\omega} = \mathbf{i}'_3 \omega, \tag{11.4}$$

in which

$$\omega = \frac{1}{2}\left(\frac{\partial u'_2}{\partial x'_1} - \frac{\partial u'_1}{\partial x'_2}\right). \tag{11.5}$$

The flow is thus characterized by an angular velocity vector possessing only a component normal to the plane of shear.

The components of the rate of strain dyadic $\mathbf{S}$ in this arbitrary system are

$$S'_{jk} = \mathbf{i}'_k \mathbf{i}'_j : \mathbf{S}. \tag{11.6}$$

In consequence of the incompressibility condition $S'_{ii} = 0$ and the symmetry condition $S'_{jk} = S'_{kj}$, these components satisfy

$$S'_{11} + S'_{22} = 0, \tag{11.7}$$

and

$$S'_{12} = S'_{21}, \tag{11.8}$$

all other strain rate components being zero. Hence, the most general two-dimensional rate of strain dyadic may be written as

$$\mathbf{S} = (\mathbf{i}'_1\mathbf{i}'_1 - \mathbf{i}'_2\mathbf{i}'_2)S'_{11} + (\mathbf{i}'_1\mathbf{i}'_2 + \mathbf{i}'_2\mathbf{i}'_1)S'_{12}, \qquad [11.9]$$

with

$$S'_{11} = \frac{\partial u'_1}{\partial x'_1}, \quad S'_{12} = \frac{1}{2}\left(\frac{\partial u'_1}{\partial x'_2} + \frac{\partial u'_2}{\partial x'_1}\right). \qquad [11.10a, b]$$

Any two-dimensional linear shear flow will therefore be of the form

$$u'_1 = S'_{11}x'_1 + (S'_{12} - \omega)x'_2, \qquad [11.11a]$$

$$u'_2 = (S'_{12} + \omega)x'_1 - S'_{11}x'_2, \qquad [11.11b]$$

$$u'_3 = 0. \qquad [11.11c]$$

The hydrodynamic properties of such a flow can thus be characterized generally by the three independent scalar parameters S'_{11}, S'_{12} and ω.

Rather than describing the flow in terms of the arbitrary system (x'_1, x'_2, x'_3) it is convenient to refer the motion to a coordinate system composed of the principal axes of **S** and the direction of the fluid angular velocity vector $\boldsymbol{\omega}$. Since **S** is symmetric, traceless and planar, it can be expressed in terms of its principal axes as

$$\mathbf{S} = (\boldsymbol{\delta}_1\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2\boldsymbol{\delta}_2)S, \qquad [11.12]$$

in which

$$S = (\tfrac{1}{2}\mathbf{S}:\mathbf{S})^{1/2} \equiv (S'^2_{11} + S'^2_{12})^{1/2}. \qquad [11.13]$$

Here, $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ are the eigenvectors of **S**, normalized to unity, and defined by

$$\mathbf{S}\cdot\boldsymbol{\delta}_1 = S_1\boldsymbol{\delta}_1, \quad \mathbf{S}\cdot\boldsymbol{\delta}_2 = S_2\boldsymbol{\delta}_2, \qquad [11.14a, b]$$

with

$$|\boldsymbol{\delta}_1| = |\boldsymbol{\delta}_2| = 1, \qquad [11.15]$$

and

$$S_1 = S, \quad S_2 = -S, \qquad [11.16a, b]$$

the eigenvalues of **S**. The "1" and "2" directions correspond, respectively, to the principal axes of tension and compression.

Since **S** is symmetric, these unit vectors are mutually perpendicular and lie in the $x'_1 - x'_2$ plane. Together with $\mathbf{i}'_3$ they form a right-handed system of mutually perpendicular unit vectors $(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\delta}_3)$, with

$$\boldsymbol{\delta}_3 \equiv \mathbf{i}'_3 \qquad [11.17]$$

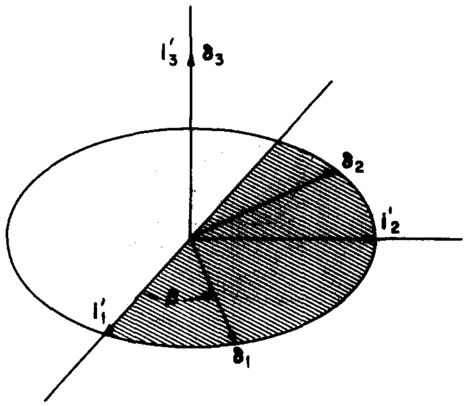

Figure 15. Orientation of the principal axes of shear of an arbitrary two-dimensional shear flow.

As in figure 15, let β denote the angle required to bring the $\mathbf{i}'_1$ axis into coincidence with the $\boldsymbol{\delta}_1$ axis via a rotation about the $\boldsymbol{\delta}_3$ axis. Since equations [11.14] are invariant under the transformations $\boldsymbol{\delta}_1 \to -\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2 \to -\boldsymbol{\delta}_2$ we may (arbitrarily) define the direction of the $\boldsymbol{\delta}_1$ axis such that β lies in the range

$$-\pi/2 < \beta \le \pi/2. \qquad [11.18]$$

Having chosen β in this manner the direction of $\boldsymbol{\delta}_2$ is then uniquely determined by the requirement that $(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\delta}_3)$, in that order, form a right-handed system.

It follows that

$$\boldsymbol{\delta}_1 = \mathbf{i}'_1 \cos\beta + \mathbf{i}'_2 \sin\beta, \qquad [11.19a]$$

$$\boldsymbol{\delta}_2 = -\mathbf{i}'_1 \sin\beta + \mathbf{i}'_2 \cos\beta. \qquad [11.19b]$$

In conjunction with [11.9] and [11.12] these show that the angle β is determined by the relations

$$\cos 2\beta = S'_{11}/S, \quad \sin 2\beta = S'_{12}/S. \qquad [11.20a, b]$$

Considered jointly with [10.26] these relations serve to establish whether $\boldsymbol{\delta}_1$ lies in the first or fourth quadrant of the $x'_1 - x'_2$ system.

Since $\boldsymbol{\Lambda} = -\boldsymbol{\varepsilon}\cdot\boldsymbol{\omega}$ (i.e. $\Lambda_{ij} = -\varepsilon_{ijk}\omega_k$) then [11.4] shows that in the principal axis system,

$$\boldsymbol{\Lambda} = (\boldsymbol{\delta}_2\boldsymbol{\delta}_1 - \boldsymbol{\delta}_1\boldsymbol{\delta}_2)\omega. \qquad [11.21]$$

Hence, from this and [11.12] the dyadic $\mathbf{H}$ defined in [10.16] may be written in the principal axis system as

$$\mathbf{H} = (\boldsymbol{\delta}_1\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2\boldsymbol{\delta}_2)BS + (\boldsymbol{\delta}_2\boldsymbol{\delta}_1 - \boldsymbol{\delta}_1\boldsymbol{\delta}_2)\omega. \qquad [11.22]$$

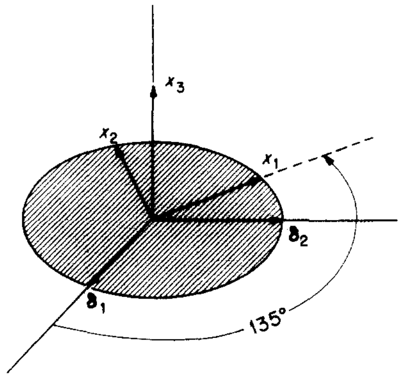

Figure 16. Orientation of the Cartesian axes x_1 and x_2 for a general two-dimensional shear flow relative to the principal axes of shear.

Now, as in figure 16, rotate the axes $(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)$ about the $\boldsymbol{\delta}_3$ axis through a positive angle of 135° to form a new system of axes (x_1, x_2) lying in the (x_1', x_2') plane perpendicular to the vorticity vector. Thus,

$$\boldsymbol{\delta}_1 = -2^{-1/2}(\mathbf{i}_1 + \mathbf{i}_2), \qquad [11.23a]$$

$$\boldsymbol{\delta}_2 = 2^{-1/2}(\mathbf{i}_1 - \mathbf{i}_2), \qquad [11.23b]$$

where $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ are a right-handed triad of mutually perpendicular unit vectors along the coordinate axes (x_1, x_2, x_3), where $x_3 \equiv x_3'$ and

$$\mathbf{i}_3 = \mathbf{i}_3' \equiv \boldsymbol{\delta}_3. \qquad [11.24]$$

In the new system it follows from [11.12] and [11.21] that

$$\mathbf{S} = (\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1)S, \qquad [11.25]$$

and

$$\boldsymbol{\Lambda} = (\mathbf{i}_2\mathbf{i}_1 - \mathbf{i}_1\mathbf{i}_2)\omega, \qquad [11.26]$$

where, in the unprimed system (cf. [11.5], [11.10] and [11.13]),

$$S = (\tfrac{1}{2}\mathbf{S}:\mathbf{S})^{1/2} = (S_{11}^2 + S_{12}^2)^{1/2}, \qquad [11.27]$$

$$\omega = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right), \quad S_{11} = \frac{\partial u_1}{\partial x_1}, \quad S_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \qquad [11.28a, b, c]$$

with

$$\mathbf{u} = \mathbf{i}_1 u_1(x_1, x_2) + \mathbf{i}_2 u_2(x_1, x_2), \qquad [11.29]$$

and

$$\boldsymbol{\omega} = \mathbf{i}_3\omega. \qquad [11.30]$$

Equations [11.25] and [11.26] combine to yield

$$\mathbf{H} = \tfrac{1}{2}B^*G^*(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1) + \tfrac{1}{2}G^*(\mathbf{i}_2\mathbf{i}_1 - \mathbf{i}_1\mathbf{i}_2), \qquad [11.31]$$

wherein we have introduced two quantities, G^* and B^*, defined in terms of the specified parameters ω, B and S as*

$$G^* = 2\omega, \qquad [11.32]$$

$$B^* = BS/\omega, \qquad [11.33]$$

provided that $\omega \neq 0$.

As a special case of this general two-dimensional flow consider the simple shear flow [8.1] taking place with respect to the (x_1, x_2, x_3) system. For such a flow we have from [11.28] that

$$S_{11} = 0, \quad S_{12} = \omega = \tfrac{1}{2}G, \qquad [11.34a, b]$$

whence, from [11.27],

$$S = \tfrac{1}{2}G, \qquad [11.35]$$

since we are supposing that $G > 0$. Under these circumstances, [11.32] and [11.33] give

$$G^* = G, \quad B^* = B, \qquad [11.36a, b]$$

whereupon [11.31] becomes

$$\mathbf{H} = \tfrac{1}{2}BG(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1) + \tfrac{1}{2}G(\mathbf{i}_2\mathbf{i}_1 - \mathbf{i}_1\mathbf{i}_2), \qquad [11.37]$$

in agreement with [10.22] for a simple shear flow.

Since the rotary velocity $\dot{\mathbf{e}}$ is given generally by [11.15] for any homogeneous shear flow, comparison of [11.31] with [11.37] shows that any linear, homogeneous, two-dimensional flow (for which $\omega \neq 0$), characterized by the flow parameters ω and S, in which are suspended identical axisymmetric particles characterized by the parameter B, can be put into a one-to-one correspondence with a simple shear flow characterized by the velocity gradient G^* in which are suspended axisymmetric particles characterized by the parameter B^*. By "one-to-one" correspondence here we mean either insofar as rotation of an isolated body in the absence of Brownian motion is concerned, or insofar as the orientational distribution function is concerned. This follows from [10.51] and [10.52], which show that two bodies possessing the same $\dot{\mathbf{e}}$ vector for a specified orientation $\mathbf{e}$ (and the same D_r) necessarily possess the same distribution function. Stated explicitly, if $f(\theta, \phi; B^*, P^* = G^*/D_r)$ represents the orientational distribution function for a simple shear flow characterized by the parameters B^* and P^*, then for the same orientation (θ, ϕ) (relative to the x_1, x_2, x_3 system), f also represents the distribution function for a two-dimensional flow characterized by the parameters B, S and ω, provided that G^* and B^* are defined as in [11.32] and [11.33].

* Since we wish to have $G^* > 0$ it is necessary to arrange matters such that $\omega > 0$. This can always be done by choosing the $\mathbf{i}'_3 \equiv \mathbf{i}_3$ direction in [11.4] or [11.30] such that $\mathbf{i}_3 \cdot \boldsymbol{\omega} > 0$. In turn, this can be done by a proper choice of the "1" and "2" directions, such that the right-handedness of the coordinates is maintained in the order 1, 2, 3.

That this is the case can perhaps be seen more explicitly as follows. Let angles (θ, ϕ) be defined relative to the (x_1, x_2, x_3) system as in figure 6. With $\mathbf{e} \equiv \mathbf{i}_r$ a unit radial vector in spherical-polar coordinates (Brenner & Condiff 1974) it is readily shown that

$$\dot{\mathbf{e}} \equiv \frac{d\mathbf{e}}{dt} = \mathbf{i}_\theta \dot{\theta} + \mathbf{i}_\phi \sin \theta \, \dot{\phi}. \tag{11.38}$$

in which $(\mathbf{i}_r, \mathbf{i}_\theta, \mathbf{i}_\phi)$ denote unit vectors in the (r, θ, ϕ) system. Use of the idemfactor in the form $\mathbf{I} = \mathbf{i}_r\mathbf{i}_r + \mathbf{i}_\theta\mathbf{i}_\theta + \mathbf{i}_\phi\mathbf{i}_\phi$, in conjunction with [10.15], yields

$$\mathbf{i}_\theta \dot{\theta} + \mathbf{i}_\phi \sin \theta \, \dot{\phi} = (\mathbf{i}_\theta\mathbf{i}_\theta + \mathbf{i}_\phi\mathbf{i}_\phi) \cdot \mathbf{H} \cdot \mathbf{i}_r.$$

Equating components gives, in general,

$$\dot{\theta} = H_{\theta r}, \quad \dot{\phi} = \frac{1}{\sin \theta} H_{\phi r}, \tag{11.39}$$

wherein

$$H_{\alpha\beta} = \mathbf{i}_\alpha \mathbf{i}_\beta : \mathbf{H}. \tag{11.40}$$

Use of the relations

$$\mathbf{i}_r = \mathbf{i}_1 \sin \theta \cos \phi + \mathbf{i}_2 \sin \theta \sin \phi + \mathbf{i}_3 \cos \theta, \tag{11.41a}$$

$$\mathbf{i}_\theta = \mathbf{i}_1 \cos \theta \cos \phi + \mathbf{i}_2 \cos \theta \sin \phi - \mathbf{i}_3 \sin \theta, \tag{11.41b}$$

$$\mathbf{i}_\phi = -\mathbf{i}_1 \sin \phi + \mathbf{i}_2 \cos \phi, \tag{11.41c}$$

connecting the spherical and Cartesian unit vectors, in conjunction with [11.31], gives rise to the relations*

$$\dot{\theta} = \tfrac{1}{4} B^* G^* \sin 2\theta \sin 2\phi, \tag{11.42}$$

$$\dot{\phi} = \tfrac{1}{2} G^* (1 + B^* \cos 2\phi). \tag{11.43}$$

These are equivalent to Jeffery's (1922) equations for a body of revolution suspended in a simple shear flow (cf. [10.1] and [10.2]).

It follows that the various moments of the distribution relative to the (x_1, x_2, x_3) system, e.g., $\langle \sin^2 \theta \rangle$, $\langle \sin^2 \theta \sin 2\phi \rangle$ and $\langle \sin^2 \theta \cos 2\phi \rangle$, are formally equivalent in the two cases.

* In addition, it is readily demonstrated from [3.3] that for a general two-dimensional linear flow the component, $\Omega_e \equiv \mathbf{e} \cdot \mathbf{\Omega}$, of the angular angular velocity vector $\mathbf{\Omega}$ for rotation of the body about its symmetry axis is given generally by the expression

$$\Omega_e = \mathbf{e} \cdot \omega = \omega \cos \theta.$$

Use of [11.32] gives

$$\Omega_e = \tfrac{1}{2} G^* \cos \theta,$$

which is the same result as for a simple shear flow at shear rate G^* (cf. [10.3]). Hence, the complete angular velocity vectors $\mathbf{\Omega}$ are identical in the two cases.

The utility of the general theorem of this section resides in the fact that it permits one to utilize the known distribution function results (including the various moments required in rheological and streaming birefringence calculations) given in Sections 8, 9 and 10 to calculate the comparable results for any two-dimensional linear shear flow. In these expressions for the moments one has only to replace B and G by $B^* \equiv BS/\omega$ and $G^* \equiv 2\omega$, respectively. Of course, one must also replace the derived parameters $P = G/D_r$, $\lambda = BG/D_r$ and $\Lambda = G(B^2 - 1)^{1/2}/D_r$ appearing in these sections by the comparable parameters

$$P^* = G^*/D_r \equiv 2\omega/D_r, \qquad [11.44]$$

$$\lambda^* = B^*G^*/D_r \equiv 2BS/D_r, \qquad [11.45]$$

$$\Lambda^* = G^*(B^{*2} - 1)^{1/2}/D_r \equiv 2(B^2S^2 - \omega^2)^{1/2}/D_r. \qquad [11.46]$$

Similarly, r_e appearing in the appropriate calculations in Sections 8, 9 and 10 pertaining to the distribution function and its moments must be replaced by r_e^*, defined as

$$r_e^* = \left(\frac{1 + B^*}{1 - B^*}\right)^{1/2} \quad (|B^*| \le 1), \qquad [11.47]$$

or

$$r_e^* = \left(\frac{B^* + 1}{B^* - 1}\right)^{1/2} \quad (|B^*| \ge 1). \qquad [11.48]$$

Of special interest is the fact that the extensive tables in Section 8, derived from the tabulations of Scheraga *et al.* (1951, 1955) and Stewart & Sørensen (1972), may be employed for these more general flows. Since these authors were only interested in bodies for which $|B| \le 1$, the tables derived from their analyses are useful in present circumstances only for situations where $|B^*| \equiv |B|S/\omega \le 1$. In this context it would prove useful to have their numerical computations extended to the case where $|B| > 1$ too.

The asymptotic analyses of Sections 9 and 10 may be employed to treat the cases where $|B^*| < 1$ and $|B^*| > 1$, respectively, in the limiting case of large Péclet numbers, for general two-dimensional shear flows.

Rheological properties of general, two-dimensional linear shear flows

Equation [4.27], which applies to all homogeneous linear flows, whether two-dimensional or not, may be written in dimensional form as

$$\mathbf{T} = 2\mu_o\mathbf{S} + \phi\mu_o[10Q_1\mathbf{S} - \tfrac{15}{2}Q_2(\mathbf{S}\cdot\langle\mathbf{ee}\rangle + \langle\mathbf{ee}\rangle\cdot\mathbf{S} - \tfrac{2}{3}\mathbf{IS}:\langle\mathbf{ee}\rangle)$$
$$- \tfrac{5}{2}B^{-1}(3Q_2 + 4Q_3^o)(\mathbf{\Lambda}\cdot\langle\mathbf{ee}\rangle - \langle\mathbf{ee}\rangle\cdot\mathbf{\Lambda}) + 5B^{-1}D_r(3Q_2 + 4Q_3)(3\langle\mathbf{ee}\rangle - \mathbf{I})]. \qquad [11.49]$$

From [11.25], [11.26], [11.32] and [11.33] the dimensional rate of strain and vorticity dyadics in the (x_1, x_2, x_3) system are

$$\mathbf{S} = \tfrac{1}{2}qG^*(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1), \quad \mathbf{\Lambda} = \tfrac{1}{2}G^*(\mathbf{i}_2\mathbf{i}_1 - \mathbf{i}_1\mathbf{i}_2), \qquad [11.50a, b]$$

where

$$q = S/\omega \equiv B^*/B. \qquad [11.50c]$$

Substitution of these into [11.49], along with use of the relations $B = q^{-1}B^*$ and [11.45], and suppression of the isotropic term $\mathbf{I}T_{33}$, yields

$$\mathbf{T} = 2\eta^* G^*[\tfrac{1}{2}(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1)] + \mathbf{i}_1\mathbf{i}_1\tau_1 + \mathbf{i}_2\mathbf{i}_2\tau_2, \qquad [11.51]$$

in which τ_1 and τ_2 are defined generally by [6.26], and η^* is defined as

$$\eta^* = T_{12}/G^*. \qquad [11.52]$$

If in these relations we define

$$[\eta^*] = \lim_{\phi \to 0} \frac{(\eta^*/q) - \mu_o}{\phi \mu_o}, \qquad [11.53]$$

and

$$[\tau_1^*] = \lim_{\phi \to 0} \frac{\tau_1}{\phi \mu_o q G^*}, \quad [\tau_2^*] = \lim_{\phi \to 0} \frac{\tau_2}{\phi \mu_o q G^*}, \qquad [11.54\text{a, b}]$$

the expressions thereby obtained for these "intrinsic" viscometric parameters are

$$[\eta^*] = 5Q_1 - \tfrac{15}{4}Q_2\langle \sin^2\theta\rangle - \tfrac{5}{4}B^{*-1}(3Q_2 + 4Q_3^o)\langle \sin^2\theta \cos 2\phi\rangle$$
$$+ \tfrac{15}{2}\lambda^{*-1}(3Q_2 + 4Q_3)\langle \sin^2\theta \sin 2\phi\rangle, \qquad [11.55]$$

$$[\tau_1^*] = 5[\tfrac{3}{4}(B^{*-1} - 1)Q_2 + B^{*-1}Q_3^o]\langle \sin^2\theta \sin 2\phi\rangle$$
$$- 15\lambda^{*-1}(3Q_2 + 4Q_3)(1 - \tfrac{3}{2}\langle \sin^2\theta\rangle - \tfrac{1}{2}\langle \sin^2\theta \cos 2\phi\rangle), \qquad [11.56]$$

$$[\tau_2^*] = -5[\tfrac{3}{4}(B^{*-1} + 1)Q_2 + B^{*-1}Q_3^o]\langle \sin^2\theta \sin 2\phi\rangle$$
$$- 15\lambda^{*-1}(3Q_2 + 4Q_3)(1 - \tfrac{3}{2}\langle \sin^2\theta\rangle + \tfrac{1}{2}\langle \sin^2\theta \cos 2\phi\rangle). \qquad [11.57]$$

The system of relations [11.51]–[11.57] become identical to the analogous simple shear relations [8.3]–[8.8] in the case where the two-dimensional flow is taken to be the simple shear flow [8.1]. This corresponds to $B^* = B$ (and, hence, $q = 1$) and $G^* = G$.

In applying these relations it must be kept in mind that they apply only in the (x_1, x_2, x_3) system, derived by rotating the principal axes of shear through 135°; that is, they apply to the coordinate system in which $\mathbf{S}$ and $\mathbf{\Lambda}$ possess the general forms set forth in [11.25] and [11.26]. Moreover, the Q values which appear in these relations are those appropriate to the value B (i.e. r_e), rather than B^* (i.e. r_e^*).

By way of a simple illustration consider the case where the rotary Brownian motion is dominant. Under these circumstances, the principal theorem of the present section shows that the goniometric factors required in [11.55]–[11.57] are given by [8.12], in which λ and B are replaced by λ^* and B^*, respectively. Hence, for $|\lambda^*| \ll 1$,

$$\langle \sin^2 \theta \rangle = \tfrac{2}{3} + \tfrac{1}{630}\lambda^{*2} + O(\lambda^{*4}), \qquad [11.58a]$$

$$\langle \sin^2 \theta \sin 2\phi \rangle = \tfrac{1}{15}[\lambda^* - \tfrac{1}{1260}(3 + 35B^{*-2})\lambda^{*3} + O(\lambda^{*5})], \qquad [11.58b]$$

$$\langle \sin^2 \theta \cos 2\phi \rangle = -\tfrac{1}{90}B^{*-1}\lambda^{*2} + O(\lambda^{*4}). \qquad [11.58c]$$

Introduction of these into [11.55]–[11.57] yields

$$[\eta^*] = [\eta]_0 - \tfrac{1}{1260}(12Q_2 + 6Q_3 + 35q^{-1}B^{*-1}N)\lambda^{*2} + O(\lambda^{*4}), \qquad [11.59a]$$

$$[\tau_1^*] = [\tfrac{1}{7}(Q_3 - Q_2) - \tfrac{1}{6}q^{-1}N]\lambda^* + O(\lambda^{*3}), \qquad [11.59b]$$

$$[\tau_2^*] = [\tfrac{1}{7}(Q_3 - Q_2) + \tfrac{1}{6}q^{-1}N]\lambda^* + O(\lambda^{*3}), \qquad [11.59c]$$

analogous to [8.13]. Here, $[\eta]_0$ is defined in [7.8].

These results can be confirmed by application of [7.4] and [7.5], which apply at small Péclet numbers to any homogeneous linear flow. These relations can be written as

$$\mathbf{T} = 2\mu_o G^*\hat{\mathbf{S}}^* + \phi\mu_o G^*[\mathbf{T}_0^* + q^{-1}\lambda^*\mathbf{T}_1^* + q^{-2}\lambda^{*2}\mathbf{T}_2^* + O(q^{-3}\lambda^{*3})], \qquad [11.60]$$

where $\mathbf{T}_0^*, \mathbf{T}_1^*, \mathbf{T}_2^*$ are the same as $\mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_2$, but with $\hat{\mathbf{S}}$ replaced by $\hat{\mathbf{S}}^*$, B replaced by $q^{-1}B^*$, and $\hat{\mathbf{\Lambda}}$ replaced by $\hat{\mathbf{\Lambda}}^*$. The quantities $\hat{\mathbf{S}}^*$ and $\hat{\mathbf{\Lambda}}^*$ represent the values of $\mathbf{S}$ and $\mathbf{\Lambda}$ in [11.50] rendered dimensionless with G^*, i.e.

$$\hat{\mathbf{S}}^* = \tfrac{1}{2}q(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1), \quad \hat{\mathbf{\Lambda}}^* = \tfrac{1}{2}(\mathbf{i}_2\mathbf{i}_1 - \mathbf{i}_1\mathbf{i}_2). \qquad [11.61a, b]$$

Use of these relations in [11.60] correctly reproduces the results cited in [11.59].

Relationship to the work of Wayland (1960)

Wayland (1960) undertakes the problem of streaming birefringence in a dilute suspension of spheroidal Brownian particles undergoing a general two-dimensional shearing flow. In this context, he calculates (for the case of dominant Brownian motion) the orientational distribution function f relative to an "intrinsic" system of Cartesian axes which translate with the fluid and maintain a fixed orientation relative to the local direction of the streamline along which the fluid translates. This contrasts with our previous calculations, which describes the (local) distribution of orientations relative to a material observer, whose orientation in space remains fixed while he translates with the fluid. It will be demonstrated in this subsection that, by an appropriate re-interpretation of the basic physical parameters characterizing the problem, the analysis can be reduced to that for a simple shear or Couette flow, for which the essentially complete analysis is already available in Sections 8–10. Moreover, the subsequent analysis applies equally well to bodies of revolution other than spheroids.

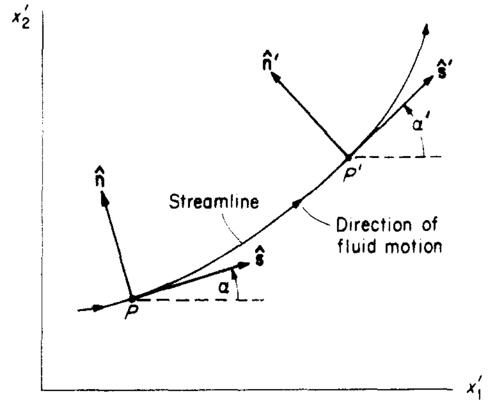

Figure 17. Curved streamlines in the plane of a two-dimensional flow.

As in figure 17, consider a typical curved streamline of the two-dimensional flow taking place in the $x_1' - x_2'$ plane, perpendicular to the vorticity vector. As stated earlier in this section, the orientation of these Cartesian axes is arbitrary. The x_3' axis is directed out of the plane of the paper. Denote by $\hat{\mathbf{s}}$ a unit tangent vector to the planar streamline at P, the sense of this vector being such as to point in the direction of motion of a material point traversing the streamline. Thus, with $\mathbf{R}$ as the position vector,

$$\hat{\mathbf{s}} = \partial \mathbf{R}/\partial s, \tag{11.62}$$

where ds is a scalar element of arc length along the streamline, taken to be positive in the direction in which $\hat{\mathbf{s}}$ points. The direction of the vector $\mathbf{i}_3' \equiv \mathbf{i}_3$ is chosen such that the scalar ω in [11.30] is non-negative. In addition, let $\hat{\mathbf{n}}$ be a unit normal vector to the streamline at P, the sense of this vector being chosen such that $(\hat{\mathbf{s}}, \hat{\mathbf{n}}, \mathbf{i}_3)$, in that order, constitute a right-handed triad of mutually perpendicular unit vectors.

In terms of these quantities the local fluid velocity vector $\mathbf{u} = D\mathbf{R}/Dt$ is given by

$$\mathbf{u} = \hat{\mathbf{s}} u(x_1', x_2'), \tag{11.63}$$

where $u \equiv |\mathbf{u}| \geq 0$ is the speed of a material point along the streamline,

$$u = \frac{Ds}{Dt} \tag{11.64}$$

the operator D/Dt being the material derivative.

The axes $(\hat{\mathbf{s}}, \hat{\mathbf{n}}, \mathbf{i}_3)$ represent an intrinsic local Cartesian coordinate system, which maintains a fixed orientation relative to the direction of the streamlines. As the material point moves along a given streamline, this system of intrinsic axes, regarded as affixed to the material point, rotates. The instantaneous orientation of the pair of axes $(\hat{\mathbf{s}}, \hat{\mathbf{n}})$ in the plane can be specified, for example, by the angle α required to bring the x_1' axis into coincidence with the direction of $\hat{\mathbf{s}}$ upon positive rotation about the $\mathbf{i}_3$ axis. More precisely, we define the angle α by the relation

$$\hat{\mathbf{s}} = \mathbf{i}'_1 \cos\alpha + \mathbf{i}'_2 \sin\alpha. \tag{11.65}$$

Since $\hat{\mathbf{n}} = \mathbf{i}_3 \times \hat{\mathbf{s}}$, this makes

$$\hat{\mathbf{n}} = -\mathbf{i}'_1 \sin\alpha + \mathbf{i}'_2 \cos\alpha. \tag{11.66}$$

The vector $\hat{\mathbf{s}}$ is locked into the intrinsic reference frame and rotates with it. Accordingly, from rigid-body kinematics (Goldstein 1950), if $\mathbf{\Gamma}$ is the angular velocity with which the intrinsic reference frame rotates relative to a space-fixed observer,

$$\frac{D\hat{\mathbf{s}}}{Dt} = \mathbf{\Gamma} \times \hat{\mathbf{s}}.$$

Inasmuch as the vector $\mathbf{\Gamma}$ possesses no component in the $\hat{\mathbf{s}}$ (or $\hat{\mathbf{n}}$) direction, the above may be solved for $\mathbf{\Gamma}$ to yield

$$\mathbf{\Gamma} = \hat{\mathbf{s}} \times \frac{D\hat{\mathbf{s}}}{Dt}. \tag{11.67}$$

Differentiation of [11.65] and subsequent use of [11.66] gives

$$\begin{aligned}\frac{D\hat{\mathbf{s}}}{Dt} &= (-\mathbf{i}'_1 \sin\alpha + \mathbf{i}'_2 \cos\alpha)\frac{D\alpha}{Dt} \\ &\equiv \hat{\mathbf{n}}\dot{\alpha},\end{aligned} \tag{11.68}$$

in which

$$\dot{\alpha} \equiv \frac{D\alpha}{Dt}. \tag{11.69}$$

Consequently, [11.67] yields

$$\mathbf{\Gamma} = \mathbf{i}_3\dot{\alpha} \tag{11.70}$$

for the angular velocity vector of the intrinsic reference frame.

In order to obtain an explicit expression for $\dot{\alpha}$, we have by the chain rule that

$$\frac{D\hat{\mathbf{s}}}{Dt} = \frac{\partial\hat{\mathbf{s}}}{\partial s}\frac{Ds}{Dt}. \tag{11.71}$$

However, by Frenet's formula (Milne-Thompson 1960)

$$\frac{\partial\hat{\mathbf{s}}}{\partial s} = \kappa_s\hat{\mathbf{n}}, \tag{11.72}$$

in which κ_s is the curvature of the streamline at P. This curvature is positive or negative, according as $\hat{\mathbf{n}}$ points from the convex to the concave side of the streamline, or conversely. Its magnitude is $|\kappa_s| = |\partial\hat{\mathbf{s}}/\partial s| \equiv |\partial^2\mathbf{R}/\partial s^2|$, which is the inverse of the radius R of curvature

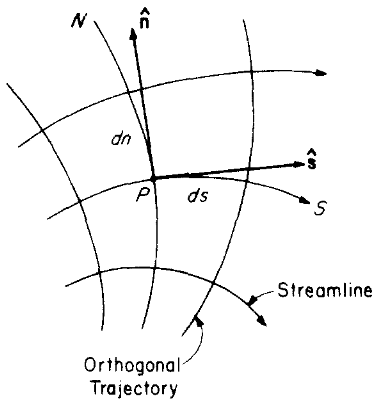

Figure 18. Curved streamlines and their orthogonal trajectories for a two-dimensional flow.

of the streamline. Equations [11.68], [11.71] and [11.72] in conjunction with [11.64] combine to yield

$$\dot{\alpha} = \kappa_s u. \qquad [11.73]$$

Consider now an intrinsic system of orthogonal curvilinear coordinates in the plane, composed of the streamlines, $\widehat{PS}$, and their orthogonal trajectories, $\widehat{PN}$, as in figure 18. The scalar dn is an element of arc length drawn along the orthogonal trajectory, and is taken to be positive when measured in the direction of $\hat{\mathbf{n}}$. In this intrinsic system the gradient operator is (Milne–Thompson 1960)

$$\nabla = \hat{\mathbf{s}}\frac{\partial}{\partial s} + \hat{\mathbf{n}}\frac{\partial}{\partial n} + \mathbf{i}_3\frac{\partial}{\partial x_3}. \qquad [11.74]$$

Since $\mathbf{u}$ is of the form [11.63] it readily follows that the local velocity gradient referred to the intrinsic axes is

$$\nabla\mathbf{u} = \hat{\mathbf{s}}\hat{\mathbf{s}}\frac{\partial u}{\partial s} + \hat{\mathbf{n}}\hat{\mathbf{n}}\kappa_n u + \hat{\mathbf{s}}\hat{\mathbf{n}}\kappa_s u + \hat{\mathbf{n}}\hat{\mathbf{s}}\frac{\partial u}{\partial n}, \qquad [11.75]$$

wherein we have employed [11.72] and its counterpart for the normal derivative (Milne–Thompson 1960),

$$\frac{\partial\hat{\mathbf{s}}}{\partial n} = \kappa_n\hat{\mathbf{n}}, \qquad [11.76]$$

with κ_n the curvature of the plane curve $\widehat{PN}$ orthogonal to the streamline at P. In consequence of the incompressibility condition $\nabla\cdot\mathbf{u} = 0$, equation [11.75] requires that

$$\frac{\partial u}{\partial s} + \kappa_n u = 0, \qquad [11.77]$$

which may be used to simplify [11.75].

In view of [11.3] and [11.30], insertion of a cross-product symbol between the antecedents and consequents of [11.75] gives (cf. Milne–Thompson 1960)

$$\omega = \frac{1}{2}\left(\kappa_s u - \frac{\partial u}{\partial n}\right). \qquad [11.78]$$

Equations [11.75] and [11.77] yield, for the rate of strain dyadic,

$$\mathbf{S} = (\hat{\mathbf{s}}\hat{\mathbf{s}} - \hat{\mathbf{n}}\hat{\mathbf{n}})\frac{\partial u}{\partial s} + (\hat{\mathbf{n}}\hat{\mathbf{s}} + \hat{\mathbf{s}}\hat{\mathbf{n}})\frac{1}{2}\left(\kappa_s u + \frac{\partial u}{\partial n}\right). \qquad [11.79]$$

For the fractional elongation rate, defined as

$$S_{ns} = \hat{\mathbf{n}} \cdot \mathbf{S} \cdot \hat{\mathbf{s}}, \qquad [11.80]$$

this gives (cf. Milne–Thompson 1960)

$$S_{ns} = \frac{1}{2}\left(\kappa_s u + \frac{\partial u}{\partial n}\right). \qquad [11.81]$$

Addition of [11.78] and [11.81] with subsequent utilization of [11.73] furnishes the relation

$$\omega = \dot{\alpha} - S_{ns}. \qquad [11.82]$$

With use of [11.12], [11.19], [11.65] and [11.66], equation [11.80] becomes

$$S_{ns} = -S \sin 2\lambda_o, \qquad [11.83]$$

in which

$$\lambda_o = \alpha - \beta. \qquad [11.84]$$

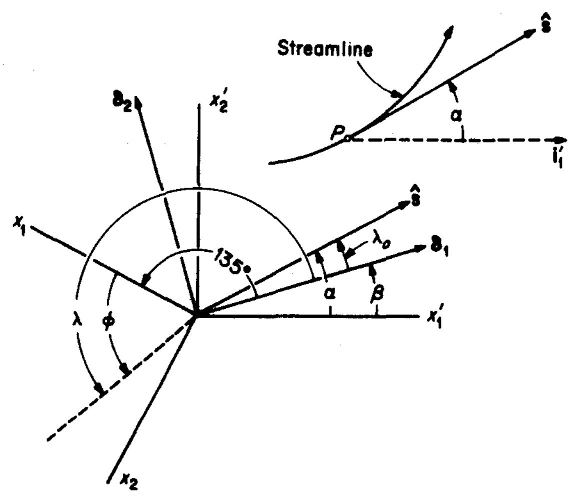

Figure 19. Relationships between different Cartesian coordinate systems pertaining to a curved streamline in a plane.

As shown in figure 19, λ_o is the angle between the direction $\boldsymbol{\delta}_1$ of the first principal strain axis and the direction $\hat{\mathbf{s}}$ of the streamline, in the sense shown in the figure. More precisely λ_o is the angle defined by the relation

$$\hat{\mathbf{s}} = \boldsymbol{\delta}_1 \cos \lambda_o + \boldsymbol{\delta}_2 \sin \lambda_o, \tag{11.85}$$

as follows from [11.65] and [11.19]. Consequently,

$$\sin 2\lambda_o = 2(\boldsymbol{\delta}_1 \cdot \hat{\mathbf{s}})(\boldsymbol{\delta}_2 \cdot \hat{\mathbf{s}}), \tag{11.86}$$

which may be used in [11.83].

From [11.42], [11.43] and [11.32], [11.33] we have for a general two-dimensional flow that

$$\frac{D\theta}{Dt} = \tfrac{1}{2} BS \sin 2\theta \sin 2\phi, \tag{11.87}$$

$$\frac{D\phi}{Dt} = \omega + BS \cos 2\phi, \tag{11.88}$$

where, as usual, θ is the polar angle measured with respect to the $\mathbf{i}_3$ axis, and ϕ is the azimuthal angle shown in figure 19. Recall that the $x_1 - x_2$ axes are defined relative to the principal axes $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ as in figure 16. Alternatively, if λ is the azimuthal angle measured relative to the first principal axis $\boldsymbol{\delta}_1$, as in figure 19, then

$$\phi = \lambda - 135^\circ. \tag{11.89}$$

In these relations $D\theta/Dt$ and $D\phi/Dt$ (or $D\lambda/Dt$) are the time rates of change of the local orientation angles θ and ϕ as measured by a material observer who translates along with the fluid, while maintaining a fixed orientation relative to a set of axes fixed in space. In setting down these relations we have implicitly utilized the fact that the center of the centrally symmetric body translates along with the fluid (cf. [3.1]); that is, we are assuming that the fundamental equations [2.9] and [2.10] apply even for nonhomogeneous flows of the type under discussion. If L is the length scale of the inhomogeneity (typically the characteristic linear dimension of the apparatus in which the flow occurs) and c is the maximum linear dimension of the suspended particle, then this condition will be met when $c/L \ll 1$. This is tantamount to supposing that all of the preceding equations apply locally, where ω and S are then the local values of the fluid angular velocity and shear. It should also be emphasized that the angle ϕ is also a local value, since ϕ is defined relative to the principal axes of shear, which vary in direction from point to point in the inhomogeneous flow.

Let $\delta\theta/\delta t$ and $\delta\phi/\delta t$ denote the rates of change of the orientation angles as measured by an "intrinsic observer" who, while translating along with the fluid, rotates in such a manner as to maintain a fixed orientation relative to the streamlines. These may be obtained in the manner indicated below.

Since the orientation vector $\mathbf{e}$ is locked into the particle, it follows from rigid-body kinematics that

$$\frac{D\mathbf{e}}{Dt} = \boldsymbol{\Omega} \times \mathbf{e}, \tag{11.90}$$

where $\mathbf{\Omega}$ is the angular velocity of the suspended particle relative to a space-fixed observer. The analogous rate of change from the vantage point of an intrinsic observer is

$$\frac{\delta \mathbf{e}}{\delta t} = (\mathbf{\Omega} - \mathbf{\Gamma}) \times \mathbf{e}, \qquad [11.91]$$

sinc $\mathbf{\Omega} - \mathbf{\Gamma}$ is the angular velocity of the particle relative to this observer. The last two relations combine to yield

$$\frac{\delta \mathbf{e}}{\delta t} = \frac{D\mathbf{e}}{Dt} - \mathbf{\Gamma} \times \mathbf{e}. \qquad [11.92]$$

From [11.41] and [11.70] in conjunction with the metrical relation (Happel & Brenner 1965)

$$\mathrm{d}\mathbf{e} = \mathbf{i}_\theta \, \mathrm{d}\theta + \mathbf{i}_\phi \sin\theta \, \mathrm{d}\phi,$$

then follow the relations

$$\frac{\delta \theta}{\delta t} = \frac{D\theta}{Dt}, \quad \frac{\delta \phi}{\delta t} = \frac{D\phi}{Dt} - \dot{\alpha}, \qquad [11.93a, b]$$

or, using [11.82], [11.83], [11.87] and [11.88],

$$\frac{\delta \theta}{\delta t} = \tfrac{1}{2} BS \sin 2\theta \sin 2\phi, \qquad [11.94]$$

$$\frac{\delta \phi}{\delta t} = BS \cos 2\phi + S \sin 2\lambda_o. \qquad [11.95]$$

These equations are identical to those given by Wayland (1960).*

The rotary flux vector relative to the intrinsic observer is

$$\mathbf{j} = f\frac{\delta \mathbf{e}}{\delta t} - D_r \frac{\partial f}{\partial \mathbf{e}}. \qquad [11.96]$$

In the steady state this vector satisfies the conservation law (Brenner & Condiff 1974)

$$\frac{\partial}{\partial \mathbf{e}} \cdot \mathbf{j} = 0, \qquad [11.97]$$

leading to a second order partial differential equation for the orientational distribution function f. Wayland (1960) succeeded in obtaining the first few terms in a series solution of this problem, valid for small values of the dimensionless parameter S/D_r (and $|B| \leq 1$) —corresponding to the case where the rotary Brownian motion dominates over the shear. However, as we now show, Wayland's problem can be reduced to the case of a simple shear flow, for which essentially complete solutions are already available over the entire range of the relevant variables.

* Notational equivalences are as follows: $S \equiv E$; $B \equiv b$; $\phi \equiv \lambda - 135°$; $\lambda_o = -\Lambda_o$; $D_r \equiv D$. Wayland's azimuthal angle λ should not be confused with our weighted Péclet number λ.

Define dimensionless quantities G_* and B_* as follows:

$$G_* = 2S \sin 2\lambda_o, \qquad [11.98a]$$

$$B_* = B/\sin 2\lambda_o. \qquad [11.98b]$$

In terms of these, [11.94] and [11.95] become

$$\frac{\delta\theta}{\delta t} = \tfrac{1}{4}B_* G_* \sin 2\theta \sin 2\phi, \qquad [11.99a]$$

$$\frac{\delta\phi}{\delta t} = \tfrac{1}{2}G_*(1 + B^* \cos 2\phi). \qquad [11.99b]$$

The form of these equations is now identical to Jeffery's equations (cf. [10.1] and [10.2]) for the rotation of a body of revolution (characterized by the rotary parameter B_*) suspended in a simple shearing flow at shear rate G_*.* Indeed, for the case where the flow is the simple shear flow [8.1] we have in [11.63] that $u = Gx_1$ and

$$\hat{\mathbf{s}} = \mathbf{i}_2, \quad \hat{\mathbf{n}} = -\mathbf{i}_1,$$

so that $\mathrm{d}s \equiv \mathrm{d}x_2$ and $\mathrm{d}n = -\mathrm{d}x_1$. With use of [11.23] and [11.85] we thereby obtain $\cos\lambda_o = \sin\lambda_o = -2^{-1/2}$, whence $\lambda_o = 225°$. This makes $\sin 2\lambda_o = 1$. Since, for the simple shear flow [8.1], $S = \frac{1}{2}G$, equations [11.98] become

$$G_* = G, \quad B_* = B,$$

as was to be expected for this case.

It is an immediate consequence of [11.99] that the differential equation governing the distribution of particle orientations (and, hence, the various moments of the distribution) relative to the intrinsic axes is identical to that for a simple shear flow characterized by the parameters B_* and G_*. All of the distribution function and momental results of Sections 8–10 may therefore be applied in present circumstances by the simple expedient of replacing B and G by B_* and G_*, respectively.

By way of example, Wayland (1960) utilized his power series solution of the intrinsic distribution function for small S/D_r to calculate the goniometric factors $\langle \sin^2\theta \cos 2\phi \rangle$ and $\langle \sin^2\theta \sin 2\phi \rangle$ required in his birefringence calculations. These results can be reproduced from our present analysis as follows. For $|\lambda_*| \ll 1$, the adaptation of [8.12] to the present class of problems yields

$$\langle \sin^2\theta \cos 2\phi \rangle = -\tfrac{1}{90}B_*^{-1}\lambda_*^2 + O(\lambda_*^4),$$

* From [3.32] the component $\boldsymbol{\Omega}\cdot\mathbf{e}$ of the angular velocity vector of the rotating particle along its symmetry axis is $\boldsymbol{\omega}\cdot\mathbf{e}$ relative to a material observer. Hence, relative to an intrinsic observer, this component is

$$\Omega_e - \Gamma_e \overset{\text{def.}}{\equiv} (\boldsymbol{\Omega} - \boldsymbol{\Gamma})\cdot\mathbf{e} = (\boldsymbol{\omega} - \boldsymbol{\Gamma})\cdot\mathbf{e}.$$

From [5.5], [11.30], [11.70], [11.82], [11.83] and [11.98a] this yields

$$\Omega_e - \Gamma_e = \tfrac{1}{2}G_* \cos\theta,$$

which is the counterpart of [10.3].

and

$$\langle \sin^2\theta \sin 2\phi \rangle = \tfrac{1}{15}\lambda_*[1 - \tfrac{1}{1260}(3 + 35B_*^{-2})\lambda_*^2 + O(\lambda_*^4)],$$

in which, from [11.98],

$$\lambda_* = B_* G_*/D_r \equiv 2BS/D_r.$$

Hence,

$$\langle \sin^2\theta \cos 2\phi \rangle = -\frac{4BS^2 \sin 2\lambda_o}{90D_r^2} + \cdots,$$

and

$$\langle \sin^2\theta \sin 2\phi \rangle = \frac{2BS}{15D_r}\left[1 - \frac{B^2S^2}{9D_r^2}\left(\frac{3}{35} + \frac{\sin^2 2\lambda_o}{B^2}\right) + \cdots\right].$$

From [11.89] we have that $\cos 2\phi \equiv -\sin 2\lambda$ and $\sin 2\phi \equiv \cos 2\lambda$. Conversion to Wayland's (1960) notation utilizing the notational equivalences set forth in the footnote on page 295 thereby yields

$$\langle \sin^2\theta \sin 2\lambda \rangle = -\frac{4bE^2 \sin 2\Lambda_o}{90D^2} + \cdots,$$

and

$$\langle \sin^2\theta \cos 2\lambda \rangle = \frac{2bE}{15D}\left[1 - \frac{b^2E^2}{9D^2}\left(\frac{3}{35} + \frac{\sin^2 2\Lambda_o}{b^2}\right) + \cdots\right],$$

in exact agreement with Wayland's equation [23].*

This calculation is, of course, purely illustrative. More generally, one can employ tables 5, 6a, 6b, 6c and 10, as well as the asymptotic results of Sections 9 and 10 to obtain the pertinent goniometric factors for any values of Wayland's parameters. In this connection it is of interest to note that Riley (1973) has recently outlined a detailed numerical scheme for the solution of Wayland's distribution function differential equation for arbitrary values of the parameter λ_*. Such a scheme is now seen to be superfluous.

* The distribution function itself, rather than its moments, can also be compared with Wayland's expression for this quantity. In making this comparison we obtain identical results for the quantities F_0, F_1, F_2 appearing in his equation (22). However, in place of his expression for F_3 we obtain

$$F_3 = \frac{b^3}{4\pi}\left\{\left[\frac{1}{64}\sin^6\theta - \left(\frac{1}{60} + \frac{\sin^2 2\Lambda_o}{18b^2}\right)\sin^2\theta\right]\cos 2\lambda - \frac{\sin 2\Lambda_o}{30b}\sin^4\theta \sin 4\lambda + \frac{1}{192}\sin^6\theta\cos 6\lambda\right\}.$$

This differs in three minor respects from that of Wayland. Since the expressions for the goniometric factors agree, these discrepancies are presumably only typographical in nature.

12. UNSTEADY STATES

Equation [4.27] is applicable only to the steady state, where $\partial f/\partial t = 0$ in [4.11]. For time-dependent orientational distributions, [4:14] must be replaced by

$$\frac{1}{D_r}\frac{\partial f}{\partial t} + \lambda \nabla_e \cdot [(B^{-1}\hat{\Lambda}\cdot \mathbf{e} + \hat{\mathbf{S}}\cdot \mathbf{e} - \hat{\mathbf{S}}\cdot \mathbf{eee})f] = \nabla_e^2 f. \qquad [12.1]$$

If this equation be multiplied by $\mathbf{ee}\,\mathrm{d}^2\mathbf{e}$ and integrated over all orientations, it readily follows that the additional term

$$-\frac{1}{2BG}\frac{\partial}{\partial t}\langle \mathbf{ee}\rangle$$

will now appear on the right-hand side of [4.26]. Hence, if we define a dimensionless time $\hat{t}$ as

$$\hat{t} = Gt, \qquad [12.2]$$

[4.26] may then be written as

$$\hat{\mathbf{S}}:\langle \mathbf{eeee}\rangle = \tfrac{1}{2}[\hat{\mathbf{S}}\cdot\langle \mathbf{ee}\rangle + \langle \mathbf{ee}\rangle\cdot\hat{\mathbf{S}}] - \lambda^{-1}(3\langle \mathbf{ee}\rangle - \mathbf{I}) - \tfrac{1}{2}B^{-1}\frac{\mathscr{D}}{\mathscr{D}\hat{t}}\langle \mathbf{ee}\rangle, \qquad [12.3]$$

in which

$$\frac{\mathscr{D}\mathbf{D}}{\mathscr{D}\hat{t}} = \frac{\partial \mathbf{D}}{\partial \hat{t}} + \mathbf{D}\cdot\hat{\Lambda} - \hat{\Lambda}\cdot\mathbf{D} \qquad [12.4]$$

denotes the dimensionless, time-dependent, Jaumann derivative of an arbitrary dyadic $\mathbf{D}$ (cf. [7.2b] for the corresponding time-independent derivative, $\mathbf{J}_*(\mathbf{D})$).

It is now assumed, as is usual, that the hydrodynamic portion of the problem of calculating the stresses in the flowing suspension is governed by the quasistatic (i.e. time-independent) Stokes' equations [2.6]–[2.8]. Thus, the sole effect of the unsteady motion is assumed to reside in the fact that the orientational distribution function f will depend explicitly upon the time. Consequently, in place of [4.27], the deviatoric stress is now given by the more general expression,

$$\frac{\mathbf{T} - 2\mu_o G\hat{\mathbf{S}}}{\phi\mu_o G} = 10Q_1\hat{\mathbf{S}} - 15Q_2\,\text{sym. tr}\,(\hat{\mathbf{S}}\cdot\langle \mathbf{ee}\rangle)$$
$$+ 5\lambda^{-1}(3Q_2 + 4Q_3)(3\langle \mathbf{ee}\rangle - \mathbf{I}) + \tfrac{5}{2}B^{-1}(3Q_2 + 4Q_3^0)\frac{\mathscr{D}}{\mathscr{D}\hat{t}}\langle \mathbf{ee}\rangle, \qquad [12.5]$$

with

$$\text{sym. tr}\,\mathbf{D} \overset{\text{def.}}{=} \tfrac{1}{2}(\mathbf{D} + \mathbf{D}^\dagger) - \tfrac{1}{3}\mathbf{I}(\mathbf{I}:\mathbf{D}) \qquad [12.6]$$

the symmetric, traceless portion of a general dyadic $\mathbf{D}$. The angular brackets continue to be defined as in [4.29], with f given by the solution of [12.1]. For time-dependent fluid motions, i.e., $\mathbf{S} = \mathbf{S}(t)$, $\Lambda \equiv \Lambda(t)$, it follows that the second orientational moment will generally be of the form

$$\langle \mathbf{ee}\rangle = \text{function}[\hat{\mathbf{S}}(\hat{t}), \hat{\Lambda}(\hat{t}); B, \lambda, \hat{t}]. \qquad [12.7]$$

Substitution into [12.5] then shows that the mean deviatoric stress $\mathbf{T}$ in the suspension will generally be time-dependent.

Owing to our quasistatic assumption, the same five material constants governing steady-state rheological behavior also govern the comparable unsteady-state behavior. Prior applications of the general theory may thereby be readily extended to a variety of nonsteady motions of interest in rheological applications. For illustrative purposes only the simplest mode of unsteady behavior is considered in this section.

Stress relaxation after cessation of steady flows (*Giesekus* 1958, *Bird et al.* 1971, *Hinch & Leal* 1973)

Attention is directed to the relaxation of stresses ensuing after an arbitrarily specified steady flow is suddenly stopped. For $t < 0$ a given state of steady motion, characterized by the constant shear and vorticity dyadics $\mathbf{S}$ and $\mathbf{\Lambda}$, is assumed to exist. At time $t = 0$ the flow is abruptly stopped and subsequently maintained in that state—whereupon $\mathbf{S} = \mathbf{\Lambda} = 0$ for all $t > 0$. Due to the anisotropic orientational distribution prevailing at $t = 0$, corresponding to that which exists in the steady flow prior to cessation of the motion, the stresses do not instantaneously vanish. Rather, they decay gradually until the distribution becomes isotropic due to the Brownian rotation.

For $t < 0$ the deviatoric stress for the steady motion is given by [4.27] or, equivalently, [12.5] with $\partial/\partial t \equiv 0$. On the other hand, for $t > 0$ the deviatoric stress is given by [12.5] as

$$\mathbf{T}^+/\phi\mu_o = \tfrac{5}{2}B^{-1}(3Q_2 + 4Q_3^o)\left(6D_r + \frac{\partial}{\partial t}\right)\langle \mathbf{ee} - \tfrac{1}{3}\mathbf{I}\rangle + 30D_rN\langle \mathbf{ee} - \tfrac{1}{3}\mathbf{I}\rangle, \qquad [12.8]$$

in which [2.36] has been employed to eliminate Q_3 in favor of Q_3^o. Since no fluid motion exists, this stress arises solely in consequence of the rotary diffusion. Upon setting $\mathbf{S} = \mathbf{\Lambda} = 0$ in [12.3] it follows that for $t \geq 0$,

$$\left(6D_r + \frac{\partial}{\partial t}\right)\langle \mathbf{ee} - \tfrac{1}{3}\mathbf{I}\rangle = 0. \qquad [12.9]$$

The angular brackets here and in [12.8] refer to the orientational moment derived from the distribution function f^+ satisfying

$$\frac{1}{D_r}\frac{\partial f^+}{\partial t} = \nabla_e^2 f^+, \qquad [12.10]$$

as follows from [12.1] in the absence of fluid motion. Accordingly, the deviatoric stress adopts the simple form

$$\mathbf{T}^+ = 30\phi\mu_o D_r N\langle \mathbf{ee} - \tfrac{1}{3}\mathbf{I}\rangle. \qquad [12.11]$$

Integration of [12.9] for $t \geq 0$ yields

$$\langle \mathbf{ee} - \tfrac{1}{3}\mathbf{I}\rangle = \langle \mathbf{ee} - \tfrac{1}{3}\mathbf{I}\rangle_o \exp(-6D_rt), \qquad [12.12]$$

in which the constant, time-independent dyadic denoted by the subscript o is the initial value of $\langle \mathbf{ee} - \frac{1}{3}\mathbf{I} \rangle$ at zero time. In turn, since $f^+ = f^-$ at $t = 0$, this is the same as the value of the goniometric dyadic appropriate to the steady-state orientational distribution prevailing for $t < 0$. In this manner the deviatoric stress for $t > 0$ is given by the expression

$$\mathbf{T}^+ = 30\phi\mu_o D_r N \langle \mathbf{ee} - \tfrac{1}{3}\mathbf{I} \rangle_o \exp(-6D_r t). \qquad [12.13]$$

The stress therefore relaxes exponentially rapidly with time, the relaxation time being $(6D_r)^{-1}$.

At time $t = 0^+$ the stress is given by the above relation with the exponential factor suppressed. In contrast, the steady-state stress, $\mathbf{T}^-$, say, is given by [12.5] with $\partial/\partial t \equiv 0$. The stress is therefore discontinuous at time $t = 0$, the values at $t = 0^-$ and $t = 0^+$ being different. The diffusive contributions to the stresses $\mathbf{T}^-$ and $\mathbf{T}^+$ are the same at $t = 0$, since the orientational distribution function and, hence, $\langle \mathbf{ee} - \frac{1}{3}\mathbf{I} \rangle$ are continuous at $t = 0$. However, the contribution of $\mathbf{S}$ and $\mathbf{\Lambda}$ to the stress vanishes abruptly at $t = 0$. Therein lies the source of the stress discontinuity.

When the stress relaxation proceeds from a previous state of steady simple shear, we find from the definitions of the various goniometric and viscometric factors in [5.5], [6.26] and [8.34] that [12.13] may be written as

$$T_{12}^+/\phi\mu_o D_r = 15N\langle \sin^2\theta \sin 2\phi \rangle_o e^{-6D_r t}, \qquad [12.14a]$$

$$\Sigma_1^+ \equiv (T_{11}^+ - T_{33}^+)/\phi\mu_o D_r = 15N\langle \sin^2\theta \cos 2\phi + 3\sin^2\theta - 2 \rangle_o e^{-6D_r t}, \qquad [12.14b]$$

and

$$\Sigma_2^+ \equiv (T_{22}^+ - T_{33}^+)/\phi\mu_o D_r = -15N\langle \sin^2\theta \cos 2\phi - 3\sin^2\theta + 2 \rangle_o e^{-6D_r t}, \qquad [12.14c]$$

all other stress components T_{ij}^+ being zero. The goniometric factors indicated by the affix o are those appropriate to a simple shear flow, available in Sections 8–10. Values of the material constant N are available for a variety of bodies in Section 3.

Long thin spheroid at large Péclet numbers

The N value appropriate to a spheroid for which $r_p \gg 1$ is given by [3.20]. When the shear rate G in the steady shear flow is sufficiently small to satisfy the inequality $P^{1/3} \gg r_p \gg 1$, we find with use of [9.5]–[9.7] (with $r_e = r_p$) that at time $t = 0^+$,

$$\frac{T_{12}^+}{\phi\mu_o D_r} = -\frac{1.5 r_p^4 P^{-1}}{\ln 2r_p - 0.5}, \qquad [12.15a]$$

$$\Sigma_1^+ = -\frac{8.772 r_p}{\ln 2r_p - 0.5}, \quad \Sigma_2^+ = \frac{6r_p^2}{\ln 2r_p - 0.5}. \qquad [12.15b, c]$$

By contrast, for the steady shear, we have from [8.37] that in the same circumstances,

$$\frac{T_{12}^- - \mu_o G}{\phi\mu_o D_r} = \frac{0.315 r_p P}{\ln 2r_p - 0.5}, \qquad [12.16a]$$

$$\Sigma_1^- = -\frac{0.25 r_p^2}{\ln 2r_p - 1.5}, \quad \Sigma_2^- = \frac{0.25 r_p^4}{\ln 2r_p - 1.5}. \qquad [12.16b, c]$$

The magnitudes of the shearing and normal stresses thereby decrease by at least an order of magnitude immediately upon cessation of the flow.

Non-interacting dumbbell

With use of [3.65b] and [8.12], the stresses appropriate to a dumbbell composed of non-interacting spheres are, for small Péclet numbers, given by the expressions

$$T_{12}^{+}/\phi\mu_o D_r = (9/20)r_p^2 P e^{-6D_r t}, \quad [12.17a]$$

$$(T_{11}^{+} - T_{33}^{+})/\phi\mu_o D_r = -(3/70)r_p^2 P^2 e^{-6D_r t}, \quad [12.17b]$$

$$(T_{22}^{+} - T_{33}^{+})/\phi\mu_o D_r = (3/28)r_p^2 P^2 e^{-6D_r t}, \quad [12.17c]$$

valid for $P \ll 1$. These agree exactly with the results of Bird *et al.* (1971), derived from a detailed, small Péclet number solution of [12.10] subject to the initial condition that $f^{+} = f^{-}$ at $t = 0$. (Notational equivalences are the same as those set forth in the footnote on p. 244.) In this connection we note that the exact values for the goniometric factors for a non-interacting dumbbell ($B = 1$) required in [12.14] are already available in table 5 over the complete range of Péclet numbers.

Two-dimensional flows

The expressions [12.14] may be applied to the two-dimensional flows discussed in Section 11. As outlined there, the goniometric factors required in [12.14] can be obtained from those available for a simple shear flow in Sections 8–10 by the simple expedient of replacing B and G by B^* and G^*, respectively, defined in [11.32] and [11.33]. The "1" and "2" directions appearing in [12.14] are those defined relative to the principal axes of shear of the two-dimensional flow by [11.23]. With this choice of directions, all T_{ij}^{+} are zero, except those appearing in [12.14].

Other examples of important unsteady flows may be found in the works of Kirkwood (1967) for "stiff" linear polymer chains, Bird *et al.* (1971) for dumbbells, and Leal & Hinch (1972) and Hinch & Leal (1973) for slightly deformed spheres and spheroids.

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APPENDIX A

The Q_{ijkl} tensor for a body of revolution

From [2.13c], [2.14c] and [2.17c] it is found that the material tensor **Q** satisfies the symmetry conditions

$$Q_{ijkl} = Q_{jikl} = Q_{ijlk} = Q_{klij}. \qquad \text{[A.1]}$$

These symmetries are the same as those arising in the Hooke's law elasticity tensor (Frederick & Chang 1965, Love 1944, Landau & Lifshitz 1959) for linearly elastic anisotropic solids; that is, if, following Frederick and Chang's (1965) notation, $\sigma_{ij} = \sigma_{ji}$ represents the stress in an elastic material, and

$$l_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}) \equiv l_{lk} \qquad \text{[A.2]}$$

represents the strain (with u_i the displacement vector), then the constitutive equation for a linearly elastic anisotropic material is

$$\sigma_{ij} = C_{ijkl} l_{kl}, \qquad \text{[A.3]}$$

in which the elasticity tensor C_{ijkl} (a material tensor) possesses the same symmetries as set forth in [A.1]. In general, such a tensor possesses 21 independent scalar components.

Following Frederick & Chang (1965) we replace [A.3] by the "engineering notation" relation,

$$\sigma_\alpha = C_{\alpha\beta} l_\beta \qquad (\alpha, \beta = 1, 2, 3, 4, 5, 6), \qquad [A.4]$$

in which

$$\sigma_1 \equiv \sigma_{11}, \quad \sigma_2 \equiv \sigma_{22}, \quad \sigma_3 \equiv \sigma_{33}, \quad \sigma_4 \equiv \sigma_{23} = \sigma_{32}, \quad \sigma_5 \equiv \sigma_{31} = \sigma_{13}, \quad \sigma_6 \equiv \sigma_{12} = \sigma_{21}, \quad [A.5]$$

along with similar relations connecting the l_β to the l_{ij}. The symmetry relations [A.1] are then summarized by the relation

$$C_{\alpha\beta} = C_{\beta\alpha}.$$

Upon writing out [A.3] and [A.4] explicitly, and utilizing the equivalences [A.5] (along with similar equivalences for the strain), we find upon comparison that

$$C_{11} = C_{1111}, \quad C_{12} = C_{1122}, \quad C_{13} = C_{1133}, \quad C_{14} = C_{1123} = C_{1132},$$
$$C_{15} = C_{1131} = C_{1113}, \quad C_{16} = C_{1112} = C_{1121},$$

and

$$C_{41} = C_{2311} = C_{3211}, \quad C_{42} = C_{2322} = C_{3222}, \quad C_{43} = C_{2333} = C_{3233},$$
$$C_{44} = 2C_{2323} = 2C_{2332} = 2C_{3223} = 2C_{3232}, \quad C_{45} = 2C_{2331} = 2C_{2313} = 2C_{3213} = 2C_{3231},$$
$$C_{46} = 2C_{2312} = 2C_{2321} = 2C_{3212} = 2C_{3221},$$

etc. In general, the factor of 2 arises only when *both* the first and second indices are either 4, 5 or 6.

Love (1944) presents a detailed investigation of the symmetry properties of the $C_{\alpha\beta}$ for the case of "transverse isotropy" (see p. 152, equation [2] and p. 154, equations [5]–[11] of Love 1944). This particular symmetry is equivalent to that of a body of revolution, though not necessarily possessing fore–aft symmetry. Using these results he demonstrates that, of the 21 independent components of $\bar{C}_{\alpha\beta}$, in a body-fixed system of Cartesian coordinates $(\bar{x}_1, \bar{x}_2\ \bar{x}_3)$, with $\bar{x}_3$ as the symmetry axis, the following components must be zero:

$$\bar{C}_{14} = \bar{C}_{15} = \bar{C}_{16} = \bar{C}_{24} = \bar{C}_{25} = \bar{C}_{26} = \bar{C}_{34} = \bar{C}_{35} = \bar{C}_{36} = \bar{C}_{45} = \bar{C}_{46} = \bar{C}_{56} = 0, \quad [A.6]$$

and that the following four relations must hold among the remaining nine nonzero components:*

$$\bar{C}_{11} = \bar{C}_{22}, \quad \bar{C}_{13} = \bar{C}_{23}, \quad \bar{C}_{44} = \bar{C}_{55}, \quad \bar{C}_{66} = \bar{C}_{11} - \bar{C}_{12}. \qquad [A.7]$$

* Love's (1944) notation differs slightly from ours, the connection between them being

$$C_{\alpha\beta}(\text{Love}) = \begin{cases} C_{\alpha\beta}(\text{this paper}) & \text{for } \beta = 1, 2, 3; \\ \tfrac{1}{2}C_{\alpha\beta}(\text{this paper}) & \text{for } \beta = 4, 5, 6. \end{cases}$$

The extra factor of 1/2 arises from the fact that in Love's (1944) definition of the strain e_{ij} we have that

$$l_{11} = e_{11}, \quad l_{22} = e_{22}, \quad l_{33} = e_{33},$$

but

$$l_{23} = \tfrac{1}{2}e_{23}, \quad l_{31} = \tfrac{1}{2}e_{31}, \quad l_{12} = \tfrac{1}{2}e_{12}.$$

Imposition of fore–aft symmetry (i.e. invariance of the body under the transformation $\bar{3} \rightarrow -\bar{3}$) does not result in any further symmetry reductions, since the existence of a center of symmetry is without effect on the general properties of the elastic coefficients (Love 1944).

We may conclude from this that the $\bar{C}_{\alpha\beta}$ matrix for a body of revolution may be characterized by only 5 independent coefficients, say $\bar{C}_{11}, \bar{C}_{12}, \bar{C}_{13}, \bar{C}_{33}$ and $\bar{C}_{44}$ (with $\bar{x}_3$ as the symmetry axis). Consequently, the only nonzero tensor components of $\bar{C}_{ijkl}$, and their relations to these 5 independent matrix coefficients, are as follows:

$$\begin{gathered}\bar{C}_{1111} = \bar{C}_{2222} \;(\equiv \bar{C}_{11}), \quad \bar{C}_{3311} = \bar{C}_{1133} = \bar{C}_{3322} = \bar{C}_{2233} \;(\equiv \bar{C}_{13}),\\ \bar{C}_{2323} = \bar{C}_{2332} = \bar{C}_{3223} = \bar{C}_{3232} = \bar{C}_{3131} = \bar{C}_{3113} = \bar{C}_{1331} = \bar{C}_{1313} \;(\equiv \tfrac{1}{2}\bar{C}_{44}),\\ \bar{C}_{2211} = \bar{C}_{1122} \;(\equiv \bar{C}_{12}), \quad \bar{C}_{3333} \;(\equiv \bar{C}_{33}),\\ \bar{C}_{1212} = \bar{C}_{1221} = \bar{C}_{2112} = \bar{C}_{2121} \;[\equiv \tfrac{1}{2}(\bar{C}_{11} - \bar{C}_{12})].\end{gathered} \qquad [A.8]$$

These relations apply, of course, only in the body-fixed system of coordinates, denoted by the overbar.

As can be verified via a term-by-term comparison, the $\bar{C}_{ijkl}$ tensor may therefore be written as

$$\begin{aligned}\bar{C}_{ijkl} = {} & \delta_{ij}\delta_{kl}C_0 + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})C_1 + (\delta_{kl}\delta_{i3}\delta_{j3} + \delta_{ij}\delta_{k3}\delta_{l3})C_2\\ & + (\delta_{jk}\delta_{i3}\delta_{l3} + \delta_{il}\delta_{j3}\delta_{k3} + \delta_{ik}\delta_{j3}\delta_{l3} + \delta_{jl}\delta_{i3}\delta_{k3})C_3\\ & + \delta_{i3}\delta_{j3}\delta_{k3}\delta_{l3}C_4,\end{aligned} \qquad [A.9]$$

in which we have defined*

$$\begin{gathered}C_0 = \bar{C}_{12}, \quad C_1 = \tfrac{1}{2}(\bar{C}_{11} - \bar{C}_{12}), \quad C_2 = \bar{C}_{13} - \bar{C}_{12},\\ C_3 = \tfrac{1}{2}(\bar{C}_{12} + \bar{C}_{44} - \bar{C}_{11}), \quad C_4 = \bar{C}_{11} + \bar{C}_{33} - 2\bar{C}_{13} - 2\bar{C}_{44}.\end{gathered}$$

With $\mathbf{e} \equiv \bar{\mathbf{1}}_3$ a unit vector drawn along the symmetry axis of the axisymmetric body, we have that the components of the vector e_i in the body-fixed frame are $(\bar{e}_1, \bar{e}_2, \bar{e}_3) = (0, 0, 1)$, i.e. $\bar{e}_i = \delta_{i3}$. Consequently, [A.9] may be written in an *arbitrary* system of Cartesian axes as

$$\begin{aligned}C_{ijkl} = {} & \delta_{ij}\delta_{kl}C_0 + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})C_1 + (\delta_{kl}e_ie_j + \delta_{ij}e_ke_l)C_2\\ & + (\delta_{jk}e_ie_l + \delta_{il}e_je_k + \delta_{ik}e_je_l + \delta_{jl}e_ie_k)C_3\\ & + e_ie_je_ke_lC_4.\end{aligned} \qquad [A.10]$$

With regard to the Q_{ijkl} tensor we may therefore write an expression identical to [A.10], with coefficients $Q_0, Q_1, \ldots, Q_4$ appearing in place of $C_0, C_1, \ldots, C_4$, respectively. Upon forming the product $Q_{ijkl}s_{kl}$ to obtain the contribution of the shear to the stresslet A_{ij} in [2.11], we observe that the contribution of the Q_0 term to A_{ij} is $\delta_{ij}Q_0s_{kk}$. This, however, is

* Inverse to these are the relations

$$\begin{gathered}\bar{C}_{11} = C_0 + 2C_1, \quad \bar{C}_{12} = C_0, \quad \bar{C}_{13} = C_0 + C_2,\\ \bar{C}_{33} = C_0 + 2C_1 + 2C_2 + 4C_3 + C_4, \quad \bar{C}_{44} = 2C_1 + 2C_3.\end{gathered}$$

identically zero in consequence of the incompressibility condition, $s_{kk} = 0$. Hence, without loss of generality we may put

$$Q_0 = 0, \qquad [A.11]$$

since the term makes no contribution to A_{ij} anyway. Furthermore, since

$$Q_{iikl}s_{kl} = (3Q_2 + 4Q_3 + Q_4)e_k e_l s_{kl}$$

(where it has been noted that $s_{kk} = 0$), it follows from the requirement of [2.15c] that the term in parentheses must be zero, i.e.

$$Q_4 = -3Q_2 - 4Q_3. \qquad [A.12]$$

Substitution of [A.11] and [A.12] into [A.10] (with the C's, of course, replaced by Q's) therefore yields the expression

$$\begin{aligned} Q_{ijkl} = {} & (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})Q_1 + (\delta_{kl}e_i e_j + \delta_{ij}e_k e_l - 3e_i e_j e_k e_l)Q_2 \\ & + (\delta_{jk}e_i e_l + \delta_{il}e_j e_k + \delta_{ik}e_j e_l + \delta_{jl}e_i e_k - 4e_i e_j e_k e_l)Q_3. \end{aligned} \qquad [A.13]$$

In consequence of this relation, the **Q** tensor may be regarded as possessing only three independent components. It should be observed that each of the three separate terms in [A.13] individually satisfies the general symmetry requirements imposed on the **Q** tensor by [2.13c], [2.14c], [2.15c] and [2.17c]. In anticipation of possible generalizations of the rheological theory to particles devoid of fore–aft symmetry, we emphasize that the form [A.13] applies without change to such circumstances. It is invariant under the transformation $\mathbf{e} \rightarrow -\mathbf{e}$, though the body geometry itself will not generally be invariant under this transformation, unless it possesses fore–aft symmetry.

An alternative and philosophically more satisfying scheme (de Groot & Mazur 1962, Jeffreys 1961) for investigating transverse isotropy utilizes the infinitesimal rotation matrix (Goldstein 1950)

$$R_{ij} \equiv \begin{bmatrix} 1 & -\mathrm{d}\phi & 0 \\ \mathrm{d}\phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for rotation about the x_3 axis to determine the symmetry restrictions resulting from the fact that the body shape is invariant under rotation through an arbitrary infinitesimal angle $\mathrm{d}\phi$ about this axis.

APPENDIX B

Material constants for axisymmetric slender bodies possessing fore–aft symmetry

Cox (1970, 1971) and Okagawa *et al.* (1973) demonstrate how the hydrodynamical resistance properties of long slender axisymmetric bodies may be calculated for the special case of simple shearing flows. It will be shown in this Appendix that the fundamental rheological material constants for such bodies may be extracted from their analyses. Once obtained, these coefficients may be applied to any type of homogeneous shearing flow.

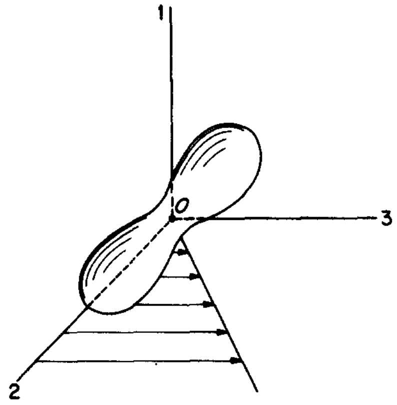

Figure 20. Axisymmetric body held in place in a simple shear flow by the action of an external couple. The streamlines are at right angles to the symmetry axis of the body.

We address ourselves first to the problem of calculating ${}^rK_\perp$ and τ. Cox (1971) considers a slender body suspended in the undisturbed simple shear flow

$$\mathbf{v}^\infty = \mathbf{i}_3 G x_2, \qquad [B.1]$$

and supposes that an external couple is exerted on the body sufficient to maintain it at rest with its symmetry axis lying in the $x_2 - x_3$ plane. When the symmetry axis coincides with the x_2 axis, perpendicular to the streamlines, as in figure 20, the couple exerted by the fluid on the body is given by Cox (1971) for $r_p \gg 1$ as $(L_1, L_2, L_3) = (L_\perp, 0, 0)$, where

$$L_\perp = \mu_o a^3 G \frac{8\pi}{3}\left[\frac{1}{\ln r_p} + \frac{K_4}{(\ln r_p)^2}\right]. \qquad [B.2]$$

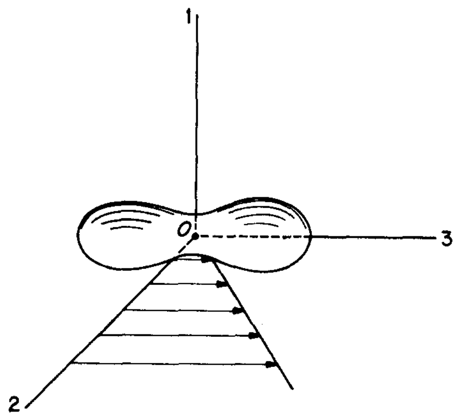

Figure 21. Axisymmetric body held in place in a simple shear flow by the action of an external couple. The streamlines are parallel to the symmetry axis of the body.

This result applies equally well to both sharp- and blunt-ended bodies.

On the other hand, when the symmetry axis of the body lies along the x_3 direction, parallel to the streamlines, as in figure 21, the corresponding couple is (Cox 1971) $(L_1, L_2, L_3) = (L_{\parallel}, 0, 0)$, in which

$$L_{\parallel} = \mu_o 2\pi ab^2 G\left[\frac{K_2}{\ln r_p} + \frac{K_3}{(\ln r_p)^2}\right] \quad [B.3]$$

for sharp-ended bodies, and

$$L_{\parallel} = \mu_o Lab^2 G \quad [B.4]$$

for blunt-ended bodies.

For a nonrotating ($\mathbf{\Omega} = 0$) axisymmetric body possessing fore–aft symmetry, we find generally from [2.10], [2.12b, c], [2.19a], [2.21] and [2.23] that

$$L_i = \mu_o 6V_p[\{e_i e_j {}^r K_{\parallel} + (\delta_{ij} - e_i e_j){}^r K_{\perp}\}\omega_j - (\varepsilon_{ijl}e_l e_k + \varepsilon_{ikl}e_l e_j)s_{jk}\tau]. \quad [B.5]$$

With the undisturbed shear flow given by [B.1] we have in the present case that

$$\omega_j = \delta_{1j}G/2, \quad [B.6]$$

and

$$s_{jk} = (\delta_{j2}\delta_{k3} + \delta_{j3}\delta_{k2})G/2. \quad [B.7]$$

When the symmetry axis of the body is oriented perpendicular to the undisturbed streamlines, as in figure 20, it follows that the components of the **e** vector in the space-fixed reference frame (x_1, x_2, x_3) are $e_m = \delta_{m2}$ $(m = 1, 2, 3)$. Consequently, it is found from [B.5]–[B.7] that $(L_1, L_2, L_3) = (L_{\perp}, 0, 0)$, with

$$L_{\perp} = 6\mu_o V_p G(\tfrac{1}{2}{}^r K_{\perp} + \tau). \quad [B.8]$$

Similarly, when the axis of revolution of the particle is oriented parallel to the streamlines, as in figure 21, $e_m = \delta_{m3}$, whence [B.5]–[B.7] yield $(L_1, L_2, L_3) = (L_{\parallel}, 0, 0)$, in which

$$L_{\parallel} = 6\mu_o V_p G(\tfrac{1}{2}{}^r K_{\perp} - \tau). \quad [B.9]$$

Since the particle volume is given by the expression $V_p = \pi ab^2 K_2$, simultaneous solution of [B.8] and [B.9] for ${}^r K_{\perp}$ and τ yields

$${}^r K_{\perp} = (6\pi\mu_o ab^2 GK_2)^{-1}(L_{\perp} + L_{\parallel}), \quad [B.10]$$

and

$$\tau = (12\pi\mu_o ab^2 GK_2)^{-1}(L_{\perp} - L_{\parallel}). \quad [B.11]$$

Substitution of [B.2]–[B.4] into these relations, and use of [2.25], yields the expressions for ${}^r K_{\perp}$ and N set forth in [3.43] and [3.44] for pointed bodies and [3.51] and [3.52] for blunt bodies.

Use of [2.28a] in conjunction with [B.10] and [B.11] gives

$$B = \frac{(L_{\perp}/L_{\parallel}) - 1}{(L_{\perp}/L_{\parallel}) + 1}. \quad [B.12]$$

This leads to the B-values set forth in [3.46] and [3.54]. As follows from [B.2]–[B.4], $L_\perp/L_\parallel \gg 1$ for both pointed and blunt bodies. Consequently, B lies in the range $0 < B < 1$ for both classes of bodies. Hence, comparison of [B.12] with [2.30] yields, for the equivalent axis ratio,

$$r_e = (L_\perp/L_\parallel)^{1/2}, \qquad \text{[B.13]}$$

in agreement with Cox (1971). This leads to the values of r_e set forth in [3.45] and [3.53].

The formula for ${}^rK_\parallel$ may be obtained by noting that the couple (L_1, L_2, L_3) exerted by the fluid on a slender axisymmetric body rotating with angular velocity $(\Omega, 0, 0)$ about its symmetry axis Ox_1 in a fluid at rest at infinity is (cf. equation [7.5] of Cox 1971)

$$L_1 = -4\mu_o\Omega V_p, \quad L_2 = L_3 = 0. \qquad \text{[B.14]}$$

However, for this case, [2.10], [2.12b] and [2.21] combine to yield

$$L_1 = -6\mu_o\Omega V_p{}^rK_\parallel, \quad L_2 = L_3 = 0. \qquad \text{[B.15]}$$

Comparison with [B.14] then gives

$${}^rK_\parallel = 2/3, \qquad \text{[B.16]}$$

valid for both sharp- and blunt-ended bodies.

Values of the translational resistance coefficients ${}^t\hat{K}_\parallel$ and ${}^t\hat{K}_\perp$ for slender bodies may be obtained as follows: for an axisymmetric body translating in a fluid at rest at infinity, [2.9] and [2.20] combine to give

$$F_\parallel = -\mu_o{}^t\hat{K}_\parallel U, \quad F_\perp = -\mu_o{}^t\hat{K}_\perp U, \qquad \text{[B.17a, b]}$$

for the forces exerted by the fluid on the body when it translates with velocity U (in a fluid at rest at infinity) parallel and perpendicular, respectively, to its symmetry axes. For these two cases, Cox (1970) gives the formulas

$$F_\parallel = -\frac{4\pi\mu_o aU}{\ln 2r_p + C_o} + O\left\{\frac{\mu_o aU}{(\ln r_p)^3}\right\}, \qquad \text{[B.18]}$$

$$F_\perp = -\frac{8\pi\mu_o aU}{\ln 2r_p + C_o + 1} + O\left\{\frac{\mu_o aU}{(\ln r_p)^3}\right\}. \qquad \text{[B.19]}$$

Comparison with [B.17] furnishes the values for the translational resistance material constants cited in [3.37] and [3.38].

The material constants Q_1, Q_2 and Q_3^o may be derived from the work of Okagawa *et al.* (1973), concerned with the rheological properties of slender axisymmetric particles suspended in a simple shear flow (in the absence of rotary Brownian motion). In essence, these authors present expressions for the A_{ij} coefficients in [3.4] for the simple shear flow [B.1] characterized by the rate of strain tensor [B.7]. In conjunction with [2.35] such information suffices to calculate the three material coefficients Q_1, Q_2 and Q_3^o.

The A_{ij} coefficients presented by Okagawa *et al.* (1973) are not precisely those appearing in [3.4]. To distinguish between the two sets of coefficients we will let $\tilde{A}_{ij}$ denote the tensor coefficients given by Okagawa, *et al.* It is readily shown* from their definition that they are related to our A_{ij} coefficients by the expression

$$A_{ij} = \frac{8\pi}{5V_p}(\tilde{A}_{ij} - \tfrac{1}{3}\delta_{ij}\tilde{A}_{kk}), \qquad \text{[B.20]}$$

where $\tilde{A}_{kk} = \tilde{A}_{11} + \tilde{A}_{22} + \tilde{A}_{33}$. Use of the expressions for the $\tilde{A}_{ij}$ (given by equations [57] and [69] of Okagawa *et al.* 1973) in the space-fixed system of Cartesian axes depicted in figure 22, then yields

$$A_{11}/G = -\frac{16C_1}{45\Lambda}(1 - 3\cos^2\theta)\sin^2\theta\sin\phi\cos\phi,$$

$$A_{22}/G = -\frac{16C_1}{45\Lambda}(1 - 3\sin^2\theta\cos^2\phi)\sin^2\theta\sin\phi\cos\phi,$$

$$A_{33}/G = -\frac{16C_1}{45\Lambda}(1 - 3\sin^2\theta\sin^2\phi)\sin^2\theta\sin\phi\cos\phi,$$

$$A_{12}/G = A_{21}/G = \frac{16C_1}{15\Lambda}\sin^3\theta\cos\theta\sin\phi\cos^2\phi,$$

$$A_{31}/G = A_{13}/G = \frac{16C_1}{15\Lambda}\sin^3\theta\cos\theta\sin^2\phi\cos\phi,$$

$$A_{23}/G = A_{32}/G = \frac{2}{5} + \frac{16C_1}{15\Lambda}\sin^4\theta\sin^2\phi\cos^2\phi, \qquad \text{[B.21]}$$

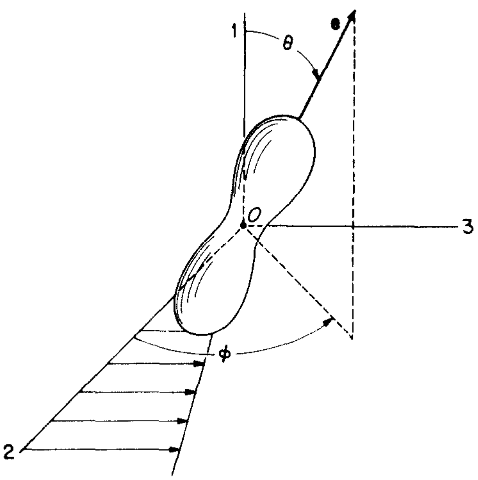

Figure 22. Definition of the orientation angles θ and ϕ used by Cox (1970, 1971) and Okagawa *et al.* (1973).

* This is perhaps most simply demonstrated by comparing equations [2.67] and [2.65] of Brenner (1972a), defining the A_{ij} in general, with equation [2.20] of Cox & Brenner (1971), defining the $\tilde{A}_{ij}$ in general.

wherein

$$C_1 = \frac{r_p^2}{4(\ln 2r_p + K)}, \qquad \text{[B.22]}$$

with K the numerical constant defined in [3.39]. In obtaining [B.21] we have utilized [3.42]. These expressions for A_{ij} apply to both pointed and blunt particles.

With use of [2.35] and [B.7], equation [3.4] may be written out in terms of the space-fixed axes of figure 22 by observing that the components of the unit vector **e** in this system are

$$e_1 = \cos\theta, \quad e_2 = \sin\theta\cos\phi, \quad e_3 = \sin\theta\sin\phi. \qquad \text{[B.23]}$$

In this manner one obtains the expressions

$$\begin{aligned}
A_{11}/G &= Q_2(1 - 3\cos^2\theta)\sin^2\theta\sin\phi\cos\phi,\\
A_{22}/G &= Q_2(1 - 3\sin^2\theta\cos^2\phi)\sin^2\theta\sin\phi\cos\phi + 2Q_3^o\sin^2\theta\sin\phi\cos\phi,\\
A_{33}/G &= Q_2(1 - 3\sin^2\theta\sin^2\phi)\sin^2\theta\sin\phi\cos\phi + 2Q_3^o\sin^2\theta\sin\phi\cos\phi,\\
A_{12}/G &= A_{21}/G = -3Q_2\sin^3\theta\cos\theta\sin\phi\cos^2\phi + Q_3^o\sin\theta\cos\theta\sin\phi,\\
A_{31}/G &= A_{13}/G = -3Q_2\sin^3\theta\cos\theta\sin^2\phi\cos\phi + Q_3^o\sin\theta\cos\theta\cos\phi,\\
A_{23}/G &= A_{32}/G = Q_1 - 3Q_2\sin^4\theta\sin^2\phi\cos^2\phi + Q_3^o\sin^2\theta.
\end{aligned} \qquad \text{[B.24]}$$

Term-by-term comparison of these general expressions with [B.21] shows that all six of these relations are satisfied by the choices

$$Q_1 = \frac{2}{5}, \quad Q_2 = -\frac{16C_1}{45\Lambda}, \quad Q_3^o = 0, \qquad \text{[B.25]}$$

leading to the results cited in [3.33]–[3.35]. Though derived by considering a simple shearing flow, the material constants [B.25] apply, of course, to any type of homogeneous shearing flow.

APPENDIX C

Material constants for dumbbells

Wakiya (1971) considers a dumbbell (see figure 2), whose center O lies at the origin of the undisturbed simple shearing flow

$$\mathbf{v}^\infty = \mathbf{i}_1 G x_3, \qquad \text{[C.1]}$$

where (x_1, x_2, x_3) constitute a system of Cartesian axes fixed in space (origin O), $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ being the corresponding unit vectors, and G the shear rate. From [2.4] and [2.2] we have for this flow that

$$\boldsymbol{\omega} = \mathbf{i}_2 G/2, \qquad \text{[C.2]}$$

and

$$\mathbf{s} = (\mathbf{i}_3\mathbf{i}_1 + \mathbf{i}_1\mathbf{i}_3)G/2. \qquad [C.3]$$

According to [2.10] and [2.19a] the couple exerted by the fluid on the dumbbell is

$$\mathbf{L} = \mu_o[{}^r\hat{\mathbf{K}} \cdot (\omega - \Omega) + \hat{\boldsymbol{\tau}}:\mathbf{s}]. \qquad [C.4]$$

Let $(\bar{\mathbf{i}}_1, \bar{\mathbf{i}}_2, \bar{\mathbf{i}}_3)$ be an orthonormal triad of right-handed unit vectors locked into the particle, with $\bar{\mathbf{i}}_3 \equiv \mathbf{e}$ lying along the symmetry axis of the dumbbell. It therefore follows from [2.21] and [2.12b] that, since

$$\bar{e}_m = \delta_{m3} \quad (m = 1, 2, 3) \qquad [C.5]$$

are the components of $\mathbf{e}$ in the body coordinates, then

$${}^rK = 16\pi c^3[\bar{\mathbf{i}}_3\bar{\mathbf{i}}_3{}^rK_{\parallel} + (\bar{\mathbf{i}}_1\bar{\mathbf{i}}_1 + \bar{\mathbf{i}}_2\bar{\mathbf{i}}_2){}^rK_{\perp}], \qquad [C.6]$$

in which it has been noted that

$$V_p = 8\pi c^3/3 \qquad [C.7]$$

is the volume of the dumbbell. In addition, [2.12c] and [2.23] in conjunction with [C.5] show that the triadic $\hat{\boldsymbol{\tau}}$ may be represented in body coordinates as

$$\hat{\boldsymbol{\tau}} = -16\pi c^3\tau(\bar{\mathbf{i}}_1\bar{\mathbf{i}}_2\bar{\mathbf{i}}_3 + \bar{\mathbf{i}}_1\bar{\mathbf{i}}_3\bar{\mathbf{i}}_2 - \bar{\mathbf{i}}_2\bar{\mathbf{i}}_1\bar{\mathbf{i}}_3 - \bar{\mathbf{i}}_2\bar{\mathbf{i}}_3\bar{\mathbf{i}}_1). \qquad [C.8]$$

In terms of body-fixed coordinates we may write

$$\Omega = \bar{\mathbf{i}}_1\bar{\Omega}_1 + \bar{\mathbf{i}}_2\bar{\Omega}_2 + \bar{\mathbf{i}}_3\bar{\Omega}_3, \qquad [C.9]$$

and

$$\mathbf{L} = \bar{\mathbf{i}}_1\bar{L}_1 + \bar{\mathbf{i}}_2\bar{L}_2 + \bar{\mathbf{i}}_3\bar{L}_3. \qquad [C.10]$$

Thus, upon performing the indicated dot multiplications in [C.4], there is obtained

$$\bar{L}_1 = -16\pi\mu_o c^3[\tau G(l_2 n_3 + l_3 n_2) + (\bar{\Omega}_1 - \tfrac{1}{2}m_1 G){}^rK_{\perp}], \qquad [C.11a]$$

$$\bar{L}_2 = 16\pi\mu_o c^3[\tau G(l_1 n_3 + l_3 n_1) - (\bar{\Omega}_2 - \tfrac{1}{2}m_2 G){}^rK_{\perp}], \qquad [C.11b]$$

$$\bar{L}_3 = -16\pi\mu_o c^3(\bar{\Omega}_3 - \tfrac{1}{2}m_3 G){}^rK_{\parallel}, \qquad [C.11c]$$

in which, for $j = 1, 2, 3$,

$$l_j = \bar{\mathbf{i}}_j \cdot \mathbf{i}_1, \quad m_j = \bar{\mathbf{i}}_j \cdot \mathbf{i}_2, \quad n_j = \bar{\mathbf{i}}_j \cdot \mathbf{i}_3 \qquad [C.12a, b, c]$$

are the direction cosines between the body-fixed and space-fixed Cartesian axes.

From equation [W-21],* Wakiya (1971) gives the relations

$$\bar{L}_1 = -16\pi\mu_o c^2 \int_0^\infty \bar{A}_{-1}\zeta \, d\zeta, \qquad [C.13a]$$

* In making reference to specific equations appearing in Wakiya's (1971) first paper, we will prefix the equation number by the letter W. For example, equation [W-21] refers to equation [21] of Wakiya (1971).

$$\bar{L}_2 = 16\pi\mu_o c^2 \int_0^\infty \bar{A}_1 \zeta \, d\zeta, \tag{C.13b}$$

$$\bar{L}_3 = 8\pi\mu_o c^2 \int_0^\infty (B_o^{-1} + B_o^{+1})\zeta \, d\zeta, \tag{C.13c}$$

where, from equation [W-17],

$$\bar{A}_1 = 2c(f_1 Y_1 - g_1 Y_2),$$
$$\bar{A}_{-1} = 2c(f_{-1} Y_1 - g_{-1} Y_2),$$

in which, from equation [W-15],

$$f_1 = Gl_1 n_3 - \bar{\Omega}_2, \quad f_{-1} = Gl_2 n_3 + \bar{\Omega}_1,$$
$$g_1 = Gl_3 n_1 + \bar{\Omega}_2, \quad g_{-1} = Gl_3 n_2 - \bar{\Omega}_1.$$

In addition, according to equation [W-16],

$$B_o^{-1} + B_o^{+1} = -2c[G(l_1 n_2 - l_2 n_1) + 2\bar{\Omega}_3]\zeta(1 - F),$$

wherein*

$$F = \tanh \zeta.$$

On substituting these results into [C.13], it follows that Wakiya's formulas for the body-fixed components of the couple are

$$\bar{L}_1 = -32\pi\mu_o c^3[G(l_2 n_3 a^2 - l_3 n_2 b^2) + \bar{\Omega}_1(a^2 + b^2)], \tag{C.14a}$$
$$\bar{L}_2 = 32\pi\mu_o c^3[G(l_1 n_3 a^2 - l_3 n_1 b^2) - \bar{\Omega}_2(a^2 + b^2)], \tag{C.14b}$$
$$\bar{L}_3 = -32\pi\mu_o c^3[\tfrac{1}{2}G(l_1 n_2 - l_2 n_1) + \bar{\Omega}_3] \int_0^\infty \zeta^2(1 - \tanh \zeta) \, d\zeta, \tag{C.14c}$$

in which a^2 and b^2 are numerical constants given by equation [W-23] as

$$a^2 = \int_0^\infty Y_1 \zeta \, d\zeta, \quad b^2 = \int_0^\infty Y_2 \zeta \, d\zeta. \tag{C.15a, b}$$

Inasmuch as the direction cosines are connected via the identities

$$m_1 = l_3 n_2 - l_2 n_3, \tag{C.16a}$$
$$m_2 = l_1 n_3 - l_3 n_1, \tag{C.16b}$$
$$m_3 = l_2 n_1 - l_1 n_2, \tag{C.16c}$$

* In contrast with the other formulas, which apply for all values of the aspect ratio r_p, this expressions for F applies only to the case where the spheres are in contact ($r_p = 1$). We shall carry along this value of F in the subsequent analysis, and later give the more general result for $\bar{L}_3$ for arbitrary r_p, kindly furnished to me by Professor Wakiya in private correspondence.

it follows that [C.11] may be written in the form

$$\bar{L}_1 = -16\pi\mu_o c^3[G\{l_2 n_3(\tfrac{1}{2}{}^rK_\perp + \tau) - l_3 n_2(\tfrac{1}{2}{}^rK_\perp - \tau)\} + \bar{\Omega}_1{}^rK_\perp], \qquad [C.17a]$$

$$\bar{L}_2 = 16\pi\mu_o c^3[G\{l_1 n_3(\tfrac{1}{2}{}^rK_\perp + \tau) - l_3 n_1(\tfrac{1}{2}{}^rK_\perp - \tau)\} - \bar{\Omega}_2{}^rK_\perp], \qquad [C.17b]$$

$$\bar{L}_3 = -16\pi\mu_o c^3[\tfrac{1}{2}G(l_1 n_2 - l_2 n_1) + \bar{\Omega}_3]{}^rK_\parallel . \qquad [C.17c]$$

Comparison of the first two of these relations with [C.14a] and [C.14b] immediately yields

$${}^rK_\perp = 2(a^2 + b^2), \qquad [C.18]$$

and

$$\tau = a^2 - b^2. \qquad [C.19]$$

Similarly, comparison of [C.14c] with [C.17c] yields (cf. footnote on page 315)

$${}^rK_\parallel = 2\int_0^\infty \zeta^2(1 - \tanh\zeta)\,d\zeta \quad \text{for } r_p = 1. \qquad [C.20]$$

In the more general case, where $r_p \neq 1$, Professor Wakiya has kindly furnished me (in private correspondence) with the following formula, derived by him:

$${}^rK_\parallel = 2\sinh^3\beta \sum_{n=1}^{\infty} n(n+1)[1 - \tanh(n + \tfrac{1}{2})\beta] \quad \text{for } r_p \neq 1, \qquad [C.21]$$

in which β is the bipolar coordinate parameter,

$$\beta = \cosh^{-1} r_p. \qquad [C.22]$$

With use of [2.25], we find from [C.19] that

$$N = (6/5)(a^2 - b^2). \qquad [C.23]$$

The B value obtained from [2.28a] is

$$B = \frac{r_e^2 - 1}{r_e^2 + 1}, \qquad [C.24]$$

in which

$$r_e = a/b \qquad [C.25]$$

is, by definition (cf. [2.30]), the equivalent axis ratio of the dumbbell. From the numerical values of a and b tabulated in table 2, it is clear that $r_e \geq 1$ for all r_p, so that $0 < B \leq 1$ for all possible values of the dumbbell aspect ratio.

It remains yet to determine the material constants Q_1, Q_2, Q_3 for the dumbbell. In polyadic notation, [3.4] is equivalent to the relation

$$\mathbf{A} = \mathbf{Q}^o\!:\!\mathbf{s}. \qquad [C.26]$$

The dyadic $\mathbf{s}$ is given for Wakiya's simple shearing flow by [C.3] in space-fixed coordinates. In body coordinates the tetradic $\mathbf{Q}^o$ is represented by the expression

$$\mathbf{Q}^o = \bar{\mathbf{i}}_i\bar{\mathbf{i}}_j\bar{\mathbf{i}}_k\bar{\mathbf{i}}_l\bar{Q}^o_{ijkl} \quad \text{(summation convention)}, \qquad [C.27]$$

in which $\bar{Q}^{o}_{ijkl}$ is given generally for a body of revolution by the right-hand side of [2.35] in which e_m is replaced by $\bar{e}_m = \delta_{m3}$.

Upon writing, in body coordinates,

$$\mathbf{A} = \bar{\mathbf{i}}_k\bar{\mathbf{i}}_l\bar{A}_{kl} \quad \text{(summation convention)}, \tag{C.28}$$

one obtains

$$\bar{A}_{ij} = Gl_k n_l \bar{Q}^{o}_{ijkl}. \tag{C.29}$$

Explicitly,

$$\bar{A}_{11} = G[(l_1n_1 - l_2n_2)Q_1 - l_3n_3(Q_1 - Q_2)], \tag{C.30a}$$

$$\bar{A}_{22} = G[(l_2n_2 - l_1n_1)Q_1 - l_3n_3(Q_1 - Q_2)], \tag{C.30b}$$

$$\bar{A}_{33} = 2Gl_3n_3(Q_1 - Q_2), \tag{C.30c}$$

$$\bar{A}_{12} = \bar{A}_{21} = G(l_1n_2 + l_2n_1)Q_1, \tag{C.30d}$$

$$\bar{A}_{23} = \bar{A}_{32} = G(l_2n_3 + l_3n_2)(Q_1 + Q_3^{o}), \tag{C.30e}$$

$$\bar{A}_{31} = \bar{A}_{13} = G(l_3n_1 + l_1n_3)(Q_1 + Q_3^{o}). \tag{C.30f}$$

In the first two of these expressions we have utilized the identity

$$l_1n_1 + l_2n_2 + l_3n_3 = 0.$$

Note that [C.30] correctly accords with the relation

$$\bar{A}_{11} + \bar{A}_{22} + \bar{A}_{33} = 0. \tag{C.31}$$

In order to obtain Wakiya's expressions for the $\bar{A}_{ij}$, we note that the stresslet $\mathbf{A}$ appearing in [2.11] is defined generally in equation [2.40] of Brenner (1972a) by the relation

$$r^5 p_{-3} = -\frac{15}{4\pi}\mu_o V_p \mathbf{A}:\mathbf{rr} \tag{C.32}$$

(subject to the conditions $A_{ij} = A_{ji}$ and $A_{ii} = 0$), in which $\mathbf{r}$ is the position vector measured from the origin O; $r = |\mathbf{r}|$, and p_{-3} is the term of $O(r^{-3})$ in the expansion of the pressure field p defined in [2.6a]. The volume V_p of the dumbbell is given by [C.7]. Wakiya (1971; equation [W-27]) writes

$$r^5 p_{-3} = -2\mu_o Q, \tag{C.33}$$

in which Q is the function

$$Q = (L - M_1)\bar{x}_1^2 + (L + M_1)\bar{x}_2^2 - 2L\bar{x}_3^2 - SM_{-1}\bar{x}_1\bar{x}_2 - 2N_1\bar{x}_3\bar{x}_1 - 2N_{-1}\bar{x}_2\bar{x}_3. \tag{C.34}$$

Thus, putting

$$\mathbf{r} = \bar{\mathbf{i}}_1\bar{x}_1 + \bar{\mathbf{i}}_2\bar{x}_2 + \bar{\mathbf{i}}_3\bar{x}_3$$

in [C.32], utilizing [C.7] and [C.28], and comparing the resulting expression with [C.33], yields

$$5c^3\bar{A}_{11} = L - M_1, \quad 5c^3\bar{A}_{22} = L + M_1, \quad 5c^3\bar{A}_{33} = -2L, \qquad \text{[C.35a, b, c]}$$

and

$$5c^3\bar{A}_{12} = 5c^3\bar{A}_{21} = -M_{-1}, \qquad \text{[C.35d]}$$

$$5c^3\bar{A}_{23} = 5c^3\bar{A}_{32} = -N_{-1}, \qquad \text{[C.35e]}$$

$$5c^3\bar{A}_{31} = 5c^3\bar{A}_{13} = N_1. \qquad \text{[C.35f]}$$

These relations accord with [C.31]. In the above, equations [W-29], [W-15] and [W-22] combine to give

$$L = -c^3 G l_3 n_3 e_0, \qquad \text{[C.36a]}$$

$$M_1 = -c^3 G(l_1 n_1 - l_2 n_2)e_2, \qquad \text{[C.36b]}$$

$$M_{-1} = -c^3 G(l_1 n_2 + l_2 n_1)e_2, \qquad \text{[C.36c]}$$

$$N_1 = -2c^3[(G l_1 n_3 - \bar{\Omega}_2)e_{11} + (G l_3 n_1 + \bar{\Omega}_2)e_{12}], \qquad \text{[C.36d]}$$

$$N_{-1} = -2c^3[(G l_2 n_3 + \bar{\Omega}_1)e_{11} + (G l_3 n_2 - \bar{\Omega}_1)e_{12}], \qquad \text{[C.36e]}$$

in which e_0, e_2, e_{11}, e_{12} are numerical constants tabulated by Wakiya (1971) as a function of r_p, and (cf. equation [W-22])

$$(a^2 + b^2)\bar{\Omega}_1 = -G(l_2 n_3 a^2 - l_3 n_2 b^2), \qquad \text{[C.37a]}$$

$$(a^2 + b^2)\bar{\Omega}_2 = G(l_1 n_3 a^2 - l_3 n_1 b^2), \qquad \text{[C.37b]}$$

$$\bar{\Omega}_3 = -\tfrac{1}{2}G(l_1 n_2 - l_2 n_1), \qquad \text{[C.37c]}$$

give the angular velocity components $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ resolved along body axes for a couple-free dumbbell (cf. [C.14] with $\bar{L}_1 = \bar{L}_2 = \bar{L}_3 = 0$).

Substitution of [C.36a, b] into [C.35a, b, c] yields

$$\bar{A}_{11} = \tfrac{1}{5}G[(l_1 n_1 - l_2 n_2)e_2 - l_3 n_3 e_0], \qquad \text{[C.38a]}$$

$$\bar{A}_{22} = \tfrac{1}{5}G[(l_2 n_2 - l_1 n_1)e_2 - l_3 n_3 e_0], \qquad \text{[C.38b]}$$

$$\bar{A}_{33} = \tfrac{1}{5}G[2 l_3 n_3 e_0]. \qquad \text{[C.38c]}$$

Comparison of [C.38a, b] with [C.30a, b] then gives $Q = \frac{1}{5}e_2$ and $Q_1 - Q_2 = \frac{1}{5}e_0$. Similarly, comparison of [C.38c] with [C.30c] also yields $Q_1 - Q_2 = \frac{1}{5}e_0$. Consequently, we have that

$$Q_1 = \tfrac{1}{5}e_2, \qquad \text{[C.39]}$$

and

$$Q_2 = \tfrac{1}{5}(e_2 - e_0). \qquad \text{[C.40]}$$

Introduction of [C.36c] into [C.35d] gives

$$\bar{A}_{12} = \bar{A}_{21} = \tfrac{1}{5}G(l_1 n_2 + l_2 n_1)e_2,$$

whereupon comparison with [C.30d] then shows that $Q_1 = \frac{1}{5}e_2$, in agreement with [C.39].

Substitution of [C.37b] into [C.36d] yields

$$N_1 = -c^3 G(l_3 n_1 + l_1 n_3)e_1,$$

in which (cf. equation [W-34]) e_1 is the numerical constant

$$e_1 \overset{\text{def.}}{=} \frac{2(e_{11} + r_e^2 e_{12})}{r_e^2 + 1}, \qquad \text{[C.41]}$$

with r_e given by [C.25]. Use of this expression for N_1 in [C.35f] then makes

$$\bar{A}_{31} = \bar{A}_{13} = \tfrac{1}{5}G(l_3 n_1 + l_1 n_3)e_1. \qquad \text{[C.42]}$$

Similarly, substitution of [C.37a] into [C.36e] gives

$$N_{-1} = -c^3 G(l_3 n_2 + l_2 n_3)e_1,$$

whence, from [C.35e],

$$\bar{A}_{23} = \bar{A}_{32} = \tfrac{1}{5}G(l_2 n_3 + l_3 n_2)e_1. \qquad \text{[C.43]}$$

Comparison of [C.42] with [C.30f], as well as [C.43] with [C.30e], furnishes the relation

$$Q_3^o = \tfrac{1}{5}(e_1 - e_2), \qquad \text{[C.44]}$$

which satisfies both pairs of relations.

The Q_3 value may now be obtained from [2.36] by employing the expressions for B and N in [C.24] and [C.23], respectively. In this manner there is obtained

$$Q_3 = \frac{1}{5}\left[e_1 - e_2 + 3b^2\frac{(r_e^2 - 1)^2}{(r_e^2 + 1)}\right]. \qquad \text{[C.45]}$$

It remains only to show that Wakiya's values for ${}^r K_{\parallel}$ in [C.20] and [C.21] agree with the accepted values (Cox & Brenner 1967) for this material constant.

Consider first the case where $r_p = 1$. Using the definitions of the hyperbolic trigonometric functions in terms of exponentials, it is easily shown that

$$^r K_{\parallel} = 2\int_0^{\infty} \zeta^2 e^{-\zeta}\,\mathrm{sech}\,\zeta\,\mathrm{d}\zeta.$$

Now, Erdélyi *et al.* (1953) give the relation

$$2^{s-1}\int_0^{\infty} t^{s-1}e^{-t}\,\mathrm{sech}\,t\,\mathrm{d}t = (1 - 2^{1-s})\Gamma(s)\zeta(s),$$

valid for $s > 0$. Here, $\zeta(s)$ is Riemann's zeta function, and $\Gamma(s)$ is the gamma function, which for s an integer is $\Gamma(s) = (s-1)!$ Choosing $s = 3$ then eventually gives

$$^r K_{\parallel} = \tfrac{3}{4}\zeta(3) = 0.90154, \qquad \text{[C.46]}$$

in accord with the results of Cox & Brenner (1967) for the tangent-sphere dumbbell.

In the case where $r_p \neq 1$, Wakiya's result [C.21] must be reconciled with the formula of Cox & Brenner (1967) for this case, cited in the last footnote on page 219. To effect this reconciliation, we note from the definition of the hyperbolic trigonometric functions that

$$1 - \tanh(n + \tfrac{1}{2})\beta = \frac{2x}{1 + x},$$

where $x = e^{-(2n+1)\beta}$. Use of the binomial expansion for $(1 + x)^{-1}$ ($|x| < 1$) then eventually gives

$$1 - \tanh(n + \tfrac{1}{2})\beta = 2 \sum_{m=1}^{\infty} (-1)^{m+1} e^{-(2n+1)m\beta},$$

whence, in [C.21],

$$\sum_{n=1}^{\infty} n(n + 1)[1 - \tanh(n + \tfrac{1}{2})\beta] = 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+1} n(n + 1) e^{-(2n+1)m\beta}.$$

The absolute convergence properties of this double sum are such that one can interchange the order of summation to obtain

$$^{r}K_{\parallel} = 4 \sinh^3 \beta \sum_{m=1}^{\infty} (-1)^{m+1} \sum_{n=1}^{\infty} n(n + 1) e^{-(2n+1)m\beta}. \qquad [C.47]$$

Now, for fixed positive values of m and β,

$$\operatorname{cosech}^3 m\beta = \frac{8e^{-3m\beta}}{(1 - e^{-2m\beta})^3} = 4 \sum_{n=1}^{\infty} n(n + 1) e^{-(2n+1)m\beta},$$

where we have employed the binomial expansion of $(1 - y)^{-3}$ ($y = e^{-2m\beta}$; $|y| < 1$). Comparison with [C.47] then yields

$$^{r}K_{\parallel} = \sinh^3 \beta \sum_{m=1}^{\infty} (-1)^{m+1} \operatorname{cosech}^3 m\beta,$$

in agreement with the formula of Cox & Brenner (1967), quoted in the last footnote on page 219.

APPENDIX D

Q *tensor for a "non-interacting" dumbbell*

As pointed out in the footnote on page 221, the Q_{ijkl} tensor given in [3.67] is not the same as the comparable tensor

$$Q'_{ijkl} = (\delta_{jk} e_i e_l + \delta_{ik} e_j e_l - \tfrac{2}{3} \delta_{ij} e_k e_l) Q, \qquad [D.1]$$

given by Brenner (1972a) for a "non-interacting" dumbbell. The apparent discrepancy stems from the fact that Q'_{ijkl} does not satisfy the (arbitrary) symmetry condition [2.13c] nor the condition [2.17a].

To put [D.1] into "standard" form we note that since $s_{kl} = s_{lk}$ and $s_{kk} = 0$, then the tensor

$$Q_{ijkl} \overset{\text{def.}}{=} \tfrac{1}{2}(Q'_{ijkl} + Q'_{ijlk}) - \tfrac{2}{3}\delta_{kl}e_i e_j Q \qquad \text{[D.2]}$$

possesses the property that

$$Q'_{ijkl}s_{kl} = Q_{ijkl}s_{kl} = Q_{klij}s_{kl}. \qquad \text{[D.3]}$$

From the point of view of [2.11] for the stresslet A_{ij}, the tensors Q_{ijkl} and Q'_{ijkl} are therefore physically equivalent.

Insertion of [D.1] into [D.2] yields

$$Q_{ijkl} = [\delta_{jk}e_i e_l + \delta_{il}e_j e_k + \delta_{ik}e_j e_l + \delta_{jl}e_i e_k - \tfrac{4}{3}(\delta_{ij}e_k e_l + \delta_{kl}e_i e_j)]\tfrac{1}{2}Q, \qquad \text{[D.4]}$$

in agreement with [3.67] to dominant terms in r_p.

APPENDIX E

Evaluation of the h-integral defined in [6.14]

Write

$$h = \frac{2}{K}J, \qquad \text{[E.1]}$$

where $J(\lambda)$ is the integral portion of [6.14]. In this integral set $2\phi = \eta$, whence

$$J = \frac{1}{2}\int_0^\pi \sin^2\theta\, G(\theta)\sin\theta\, d\theta, \qquad \text{[E.2]}$$

in which

$$G(\theta) = \int_{\eta=0}^{4\pi} \exp(\alpha\cos\eta)\, d\eta, \qquad \text{[E.3]}$$

with $\alpha(\theta) = -\frac{1}{4}\lambda\sin^2\theta$. As is readily shown,

$$G(\theta) = 2\left[\int_0^\pi \exp(\alpha\cos\eta)\, d\eta + \int_0^\pi \exp(-\alpha\cos\eta)\, d\eta\right].$$

But (McLachlan 1955)

$$\int_0^\pi \exp(\pm\alpha\cos\eta)\, d\eta = \pi I_o(\pm\alpha),$$

where I_o is the modified Bessel function of order zero. Since I_o is an even function of its argument,

$$G = 4\pi I_o(|\alpha|).$$

Consequently, from [E.2],

$$J = 2\pi \int_0^\pi \sin^2\theta\, I_o(\tfrac{1}{4}|\lambda|\sin^2\theta)\sin\theta\,d\theta$$

$$\equiv 4\pi \int_0^{\pi/2} \sin^2\theta\, I_o(\tfrac{1}{4}|\lambda|\sin^2\theta)\sin\theta\,d\theta.$$

Now set $\sin^2\theta = \cos x$. This is equivalent to $\cos\theta = 2^{1/2}\sin(x/2)$. Differentiation of both sides of the latter then gives $\sin\theta\,d\theta = -2^{-1/2}\cos(x/2)\,dx$. Therefore,

$$J = \frac{4\pi}{2^{1/2}} \int_{x=0}^{\pi/2} \cos(\tfrac{1}{2}x)\cos x\, I_o(\tfrac{1}{4}|\lambda|\cos x)\,dx.$$

Use of the trigonometric identity

$$\cos(\tfrac{1}{2}x)\cos x = \tfrac{1}{2}[\cos(\tfrac{1}{2}x) + \cos(\tfrac{3}{2}x)],$$

yields

$$J = \frac{2\pi}{2^{1/2}}(J_1 + J_2),$$

in which

$$J_1 = \int_0^{\pi/2} \cos(\tfrac{1}{2}x) I_o(\tfrac{1}{4}|\lambda|\cos x)\,dx,$$

and

$$J_2 = \int_0^{\pi/2} \cos(\tfrac{3}{2}x) I_o(\tfrac{1}{4}|\lambda|\cos x)\,dx.$$

Each of these J_i integrals may now be evaluated by application of the general theorem (Gradshteyn & Ryzhik 1965)

$$\int_0^{\pi/2} \cos(2\mu x) I_{2v}(2y\cos x)\,dx = \frac{\pi}{2}\, I_{v+\mu}(y) I_{v-\mu}(y),$$

valid for $v > -1/2$, by putting $v = 0$, $y = |\lambda|/8$, and successively choosing $\mu = 1/4$ and $3/4$. This procedure ultimately leads to

$$J = 2^{-1/2}\pi^2[I_{1/4}(\tfrac{1}{8}|\lambda|)I_{-1/4}(\tfrac{1}{8}|\lambda|) + I_{3/4}(\tfrac{1}{8}|\lambda|)I_{-3/4}(\tfrac{1}{8}|\lambda|)].$$

Use of this relation and [6.6] in [E.1] then furnishes the expression for h appearing in [6.16].

APPENDIX F

Estimates of the a, b and c coefficients in [9.13]

Stewart–Sørensen (1972) *estimate:*

Equations [9.16] indicate a $P^{-1/3}$ dependence of the three viscometric functions for a simple shearing flow at large Péclet numbers. This theoretical fact accords with the empirical numerical findings of Stewart & Sørensen (1972) for dumbbells of large aspect ratio. In turn, this information may be utilized in an indirect way to obtain estimates for the a, b, c coefficients in [9.13].

These authors, in effect, solved [8.19]–[8.21] with $B = 1$ (cf. [8.23]) by a numerical scheme, these results then being utilized to implicitly compute the three goniometric factors, $\langle \sin^2 \theta \rangle$, $\langle \sin^2 \theta \cos 2\phi \rangle$, $\langle \sin^2 \theta \sin 2\phi \rangle$, at various Péclet numbers, up to $P = 600$. These, in turn, were employed to calculate values of the three viscometric functions in [8.16a]–[8.16c]. At the larger Péclet numbers, they found by inspection that their numerical results could be accurately fitted by the equations

$$\frac{\eta - \eta_s}{n_o kT\lambda_h} = 0.678\left(\frac{1-h}{1-2h}\right)\left(\frac{P}{6}\right)^{-1/3}, \qquad \text{[F.1a]}$$

$$\frac{\beta}{n_o kT\lambda_h^2} = 0.93\left(\frac{h}{1-2h}\right)\left(\frac{P}{6}\right)^{-7/3}, \qquad \text{[F.1b]}$$

$$\frac{\Theta}{n_o kT\lambda_h^2} = 1.20\left(\frac{1-h}{1-2h}\right)\left(\frac{P}{6}\right)^{-4/3}. \qquad \text{[F.1c]}$$

If equations [9.13] are substituted into [8.16], there is obtained, for $P \gg 1$,

$$\frac{\eta - \eta_s}{n_o kT\lambda_h} \sim (b - a)\frac{3}{6^{1/3}2}\left(\frac{1-h}{1-2h}\right)\left(\frac{P}{6}\right)^{-1/3}, \qquad \text{[F.2a]}$$

$$\frac{\beta}{n_o kT\lambda_h^2} \sim (3a - b)\frac{3}{6^{1/3}2}\left(\frac{h}{1-2h}\right)\left(\frac{P}{6}\right)^{-7/3}, \qquad \text{[F.2b]}$$

$$\frac{\Theta}{n_o kT\lambda_h^2} \sim c\frac{3}{6^{1/3}}\left(\frac{1-h}{1-2h}\right)\left(\frac{P}{6}\right)^{-4/3}. \qquad \text{[F.2c]}$$

Comparison of these with [F.1] yields

$$b - a = (0.678)(2)6^{1/3}/3 = 0.822,$$

$$3a - b = (0.93)(2)6^{1/3}/3 = 1.125,$$

$$c = (1.20)6^{1/3}/3 = 0.727.$$

The numerical values of the coefficients obtained in this manner are

$$a = 0.974, \qquad b = 1.796, \qquad c = 0.727. \qquad \text{[F.3]}$$

Schwarz (1956) *estimate:*

Schwarz defines three functions $F_1(P)$, $F_2(P)$, $F_3(P)$, which in our notation (Schwarz's symbol σ is equivalent to our P) are equivalent to

$$F_1 = \tfrac{3}{2}\langle \sin^2 \theta \rangle - 1,$$

$$F_2 = \tfrac{3}{2}\langle \sin^2 \theta \cos 2\phi \rangle,$$

$$F_3 = \tfrac{3}{2}\langle \sin^2\theta \sin 2\phi\rangle.$$

For B near unity, Schwarz demonstrates, by means of an approximate procedure for determining the orientational moments of [8.19]–[8.21], that

$$F_1 \approx \tfrac{1}{2},$$

$$F_2 \approx -B\left(\frac{3}{2} - \frac{1.83}{P^{1/3}} + \frac{0.85}{P}\right),$$

$$F_3 \approx B\left(\frac{1.06}{P^{1/3}} - \frac{0.67}{P}\right).$$

These lead to the values

$$\langle \sin^2\theta\rangle \approx 1,$$

$$\langle \sin^2\theta \cos 2\phi\rangle \approx -1 + \frac{1.22}{P^{1/3}} - \frac{0.57}{P},$$

$$\langle \sin^2\theta \sin 2\phi\rangle \approx \frac{0.71}{P^{1/3}} - \frac{0.45}{P},$$

and thence to the values of a, b, c noted in [9.16]. Because of the approximate nature of Schwarz's scheme for obtaining the various moments, these values of a, b and c must be regarded as approximations, rather than rigorous estimates.

APPENDIX G

Energy dissipation

Additional energy dissipation rate. Consider an isolated solid particle of arbitrary shape undergoing translational and rotational motion in a fluid subject to a homogeneous shearing flow at infinity. The fluid motion will be assumed to be governed by the quasistatic creeping motion equations [2.6]. Let S_p denote the surface of the particle, and S_∞ the surface of a large spherical envelope of fluid containing the particle in its interior. In view of [2.7], [2.8] and [2.5], the boundary conditions are

$$\mathbf{v} = \mathbf{U} + \mathbf{\Omega} \times \mathbf{r} \quad \text{on } S_p, \qquad \text{[G.1]}$$

and

$$\mathbf{v} = \mathbf{v}^\infty \equiv \mathbf{v}^o + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{s}\cdot\mathbf{r} \quad \text{on } S_\infty, \qquad \text{[G.2]}$$

with $\mathbf{r}$ the position vector measured from the origin O.

In creeping flow the time rate E at which mechanical energy is being dissipated in the fluid region external to the particle is equal to the rate at which the stresses acting over the surfaces bounding the fluid are doing work upon it. Hence,

$$E = -\int_{S_p + S_\infty} d\mathbf{S} \cdot \mathbf{P} \cdot \mathbf{v}, \qquad [G.3]$$

where

$$\mathbf{P} = -\mathbf{I}p + \mu_o[\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger] \qquad [G.4]$$

is the pressure tensor arising from the motion [2.6] satisfying boundary conditions [G.1] and [G.2]. The directed element of surface area $d\mathbf{S}$ is drawn parallel to the inner normal to the fluid volume bounded internally by S_p and externally by S_∞. Thus, on S_p, $d\mathbf{S}$ is directed outward from the particle, into the fluid, whereas on S_∞, $d\mathbf{S}$ is directed inward, towards the particle.

In the absence of the particle, the undisturbed rate of mechanical energy dissipation resulting from the homogeneous shear flow is likewise

$$E^\infty = -\int_{S_\infty} d\mathbf{S} \cdot \mathbf{P}^\infty \cdot \mathbf{v}^\infty, \qquad [G.5]$$

in which $\mathbf{P}^\infty = 2\mu_o\mathbf{s}$ is the undisturbed pressure tensor.

The additional rate E^+ at which mechanical energy is dissipated in the fluid due to the presence of the suspended particle in the shearing flow is, therefore,

$$E^+ = E - E^\infty. \qquad [G.6]$$

Since $\mathbf{v} = \mathbf{v}^\infty$ on S_∞, then, with the aid of the reciprocal theorem (Brenner 1963)

$$\int_{S_p + S_\infty} d\mathbf{S} \cdot \mathbf{P} \cdot \mathbf{v}^\infty = \int_{S_p + S_\infty} d\mathbf{S} \cdot \mathbf{P}^\infty \cdot \mathbf{v},$$

and the relation

$$\int_{S_p} d\mathbf{S} \cdot \mathbf{P}^\infty \cdot \mathbf{v} = 0, \qquad [G.7]$$

we eventually obtain (Brenner 1958)

$$E^+ = \int_{S_p} d\mathbf{S} \cdot \mathbf{P} \cdot (\mathbf{v}^\infty - \mathbf{v}). \qquad [G.8]$$

Equation [G.7] may be proved by utilizing the boundary condition [G.1] to obtain

$$\int_{S_p} d\mathbf{S} \cdot \mathbf{P}^\infty \cdot \mathbf{v} = \mathbf{F}^\infty \cdot \mathbf{U} + \mathbf{L}^\infty \cdot \boldsymbol{\Omega}, \qquad [G.9]$$

in which

$$\mathbf{F}^\infty = \int_{S_p} d\mathbf{S} \cdot \mathbf{P}^\infty, \quad \mathbf{L}^\infty = \int_{S_p} \mathbf{r} \times (d\mathbf{S} \cdot \mathbf{P}^\infty).$$

Since $\mathbf{P}^\infty$ possesses no singularities in the interior V_p of the volume presently occupied by the particle, one may employ the divergence theorem to obtain

$$\mathbf{F}^\infty = \int_{V_p} \nabla \cdot \mathbf{P}^\infty \, dV = 0,$$

and

$$\mathbf{L}^\infty = -\int_{S_p} d\mathbf{S} \cdot (\mathbf{P}^\infty \times \mathbf{r}) = -\int_{V_p} \nabla \cdot (\mathbf{P}^\infty \times \mathbf{r}) \, dV = -\int_{V_p} (\nabla \cdot \mathbf{P}^\infty) \times \mathbf{r} \, dV = 0.$$

The vanishing of these volume integrals is a consequence of the relation

$$\nabla \cdot \mathbf{P}^\infty = 0.$$

Equation [G.9] then shows that the integral in question is zero, thereby demonstrating the validity of [G.7].

Positivity of the additional energy dissipation rate. It will now be demonstrated that the additional rate of mechanical energy dissipation satisfies the inequality,

$$E^+ \geq 2\mu_o V_p \mathbf{s}:\mathbf{s} \geq 0, \qquad \text{[G.10]}$$

in which the equality sign holds only if, simultaneously, the following conditions obtain:

$$\mathbf{U} = \mathbf{v}^o, \quad \mathbf{\Omega} = \boldsymbol{\omega}, \quad \mathbf{s} = 0, \qquad \text{[G.11a, b, c]}$$

in which case $E^+ = 0$.

In order to prove this relation, define the "additional" velocity and stress fields,

$$\mathbf{v}^+ = \mathbf{v} - \mathbf{v}^\infty, \quad \mathbf{P}^+ = \mathbf{P} - \mathbf{P}^\infty, \qquad \text{[G.12a, b]}$$

arising from the presence of the particle in the undisturbed shear flow $(\mathbf{v}^\infty, \mathbf{P}^\infty)$. With the directed elements of surface area on S_p and S_∞ defined in the manner described following [G.4], consider the integral

$$I^+ = -\int_{S_p + S_\infty} d\mathbf{S} \cdot \mathbf{P}^+ \cdot \mathbf{v}^+. \qquad \text{[G.13]}$$

It will now be demonstrated that

$$I^+ \geq 0, \qquad \text{[G.14]}$$

in which the equality sign applies only if $\mathbf{v}^+ = 0$ everywhere in the fluid volume V_f bounded internally by S_p and externally by S_∞; that is, only if $\mathbf{v} = \mathbf{v}^\infty$ for all $\mathbf{r} \in V_f$. From [G.1] and [G.2] this occurs if, and only if, conditions [G.11] are each satisfied.

To prove [G.14] we note that by the divergence theorem, [G.13] can be converted into the volume integral

$$I^+ = \int_{V_f} \nabla \cdot (\mathbf{P}^+ \cdot \mathbf{v}^+) \, dV. \qquad \text{[G.15]}$$

By identity, in Cartesian tensor notation,

$$\nabla \cdot (\mathbf{P}^+ \cdot \mathbf{v}^+) \equiv (P_{ij}^+ v_j^+)_{,i} = P_{ij,i}^+ v_j^+ + P_{ij}^+ v_{j,i}^+.$$

However, in creeping flow, $P_{ij,i} = 0$. Moreover, $P_{ij,i}^\infty = 0$. Consequently, from [G.12b],

$$P_{ij,i}^+ = 0.$$

Furthermore,

$$P_{ij}^+ = -\delta_{ij} p^+ + \mu_o (v_{j,i}^+ + v_{i,j}^+).$$

Since $\delta_{ij} v_{j,i}^+ = v_{i,i}^+ = 0$ in consequence of incompressibility, we thereby obtain

$$\nabla \cdot (\mathbf{P}^+ \cdot \mathbf{v}^+) = \mu_o (v_{j,i}^+ + v_{i,j}^+) v_{j,i}^+.$$

The tensor $v_{j,i}^+$ may be decomposed into symmetric and antisymmetric parts as follows:

$$v_{j,i}^+ = \tfrac{1}{2}(v_{j,i}^+ + v_{i,j}^+) + \tfrac{1}{2}(v_{j,i}^+ - v_{i,j}^+).$$

However, it is readily demonstrated that

$$(v_{j,i}^+ + v_{i,j}^+)(v_{j,i}^+ - v_{i,j}^+) = 0,$$

whereupon we find that

$$\nabla \cdot (\mathbf{P}^+ \cdot \mathbf{v}^+) = \tfrac{1}{2}\mu_o (v_{j,i}^+ + v_{i,j}^+)^2.$$

Since $\mu_o > 0$, it may be concluded that

$$\nabla \cdot (\mathbf{P}^+ \cdot \mathbf{v}^+) \geq 0 \quad \text{for all } \mathbf{r} \in V_f, \qquad [G.16]$$

wherein equality holds only if

$$v_{j,i}^+ + v_{i,j}^+ = 0. \qquad [G.17]$$

Substitution of [G.16] into [G.15] yields the inequality [G.14], the equality sign applying if, and only if, [G.17] holds at each point $\mathbf{r} \in V_f$. Equation [G.17] corresponds to a rigid-body motion for which

$$\mathbf{v}^+ = \mathbf{a} + \mathbf{b} \times \mathbf{r} \quad \text{for all } \mathbf{r} \in V_f,$$

with **a** and **b** constant vectors. However, [G.2] requires that

$$\mathbf{v}^+ = \mathbf{v} - \mathbf{v}^\infty \to 0 \quad \text{as } |\mathbf{r}| \to \infty.$$

Therefore, **a** and **b** must be identically zero, whence the equality sign in [G.14] applies only if

$$\mathbf{v}^+ = 0 \quad \text{for all } \mathbf{r} \in V_f. \qquad [G.18]$$

However, from [G.1], [G.2] and [G.12a], on the particle surface, $\mathbf{v}^+$ is required to satisfy the boundary condition

$$\mathbf{v}^+ = (\mathbf{U} - \mathbf{v}^o) + (\mathbf{\Omega} - \boldsymbol{\omega}) \times \mathbf{r} - \mathbf{s} \cdot \mathbf{r} \quad \text{on } S_p.$$

Thus, [G.18] will be true if, and only if, conditions [G.11] are each satisfied.

We have therefore succeeded in proving that [G.14] is valid, and that the equals sign holds only if conditions [G.11a, b, c] simultaneously obtain.

Equations [G.13], [G.14] and [G.12] combine to give

$$\int_{S_p+S_\infty} \mathbf{dS}\cdot(\mathbf{P}-\mathbf{P}^\infty)\cdot(\mathbf{v}^\infty-\mathbf{v}) \geq 0.$$

Since $\mathbf{v}^\infty - \mathbf{v} = 0$ on S_∞, the integral over S_∞ vanishes, whereupon the above inequality reduces to

$$\int_{S_p} \mathbf{dS}\cdot\mathbf{P}\cdot(\mathbf{v}^\infty-\mathbf{v}) \geq \int_{S_p} \mathbf{dS}\cdot\mathbf{P}^\infty\cdot\mathbf{v}^\infty - \int_{S_p} \mathbf{dS}\cdot\mathbf{P}^\infty\cdot\mathbf{v}.$$

Comparison with [G.8] reveals that the integral appearing on the left is E^+. Moreover, the last integral on the right is zero in consequence of [G.7]. Hence,

$$E^+ \geq \int_{S_p} \mathbf{dS}\cdot\mathbf{P}^\infty\cdot\mathbf{v}^\infty,$$

wherein the equality sign applies only if conditions [G.11] are met. By the divergence theorem, the integral on the right may be converted into a volume integral over the volume V_p of fluid presently occupied by the particle, yielding

$$\int_{S_p} d\mathbf{S}\cdot\mathbf{P}^\infty\cdot\mathbf{v}^\infty = \int_{V_p} \nabla\cdot(\mathbf{P}^\infty\cdot\mathbf{v}^\infty)\,dV.$$

As in the derivation following [G.15], it is readily shown that

$$\nabla\cdot(\mathbf{P}^\infty\cdot\mathbf{v}^\infty) = \tfrac{1}{2}\mu_o(v^\infty_{j,i}+v^\infty_{i,j})^2 = \tfrac{1}{2}\mu_o(2s_{ij})^2 \equiv 2\mu_o\mathbf{s}:\mathbf{s},$$

which is constant throughout the volume V_p. In this manner we find that

$$E^+ \geq 2\mu_o V_p\mathbf{s}:\mathbf{s}. \quad [G.19]$$

Inasmuch as $\mathbf{s}:\mathbf{s} \geq 0$ (with equality holding only when $\mathbf{s} = 0$), we may conclude from [G.19] that

$$E^+ > 2\mu_o V_p\mathbf{s}:\mathbf{s} > 0 \quad \text{unless [G.11] holds,} \quad [G.20a]$$

and that

$$E^+ = 0 \quad \text{when [G.11] holds.} \quad [G.20b]$$

From [2.9] to [2.11] it is clear that the conditions [G.11] arise when the force, torque and stresslet exerted by the fluid on the particle are identically zero, i.e.

$$\mathbf{F} = 0, \quad \mathbf{L} = 0, \quad \mathbf{A} = 0. \quad [G.21a, b, c]$$

Inequalities imposed on material constants. Introduction into [G.8] of [G.1] and the expression for $\mathbf{v}^\infty$ given in [G.2], yields

$$E^+ = \mathbf{F}\cdot(\mathbf{v}^o-\mathbf{U}) + \mathbf{L}\cdot(\boldsymbol{\omega}-\boldsymbol{\Omega}) + \left(\int_{S_p} \mathbf{dS}\cdot\mathbf{Pr}\right):\mathbf{s}, \quad [G.22]$$

in which

$$\mathbf{F} = \int_{S_p} d\mathbf{S}\cdot\mathbf{P}, \quad \mathbf{L} = \int_{S_p} \mathbf{r}\times(d\mathbf{S}\cdot\mathbf{P}), \tag{G.23a, b}$$

are, respectively, the hydrodynamic force and torque (about O) exerted by the fluid on the particle. Now, the general definition of the symmetric, traceless, dyadic "stresslet" **A** appearing in [2.11] is (Brenner 1972a)

$$\mathbf{A} = \frac{1}{5\mu_o V_p}\left[\frac{1}{2}\int_{S_p} (\mathbf{r}\,d\mathbf{S}\cdot\mathbf{P} + d\mathbf{S}\cdot\mathbf{P}\mathbf{r}) - \tfrac{1}{3}\mathbf{I}\int_{S_p} \mathbf{r}\cdot(d\mathbf{S}\cdot\mathbf{P})\right]. \tag{G.24}$$

Since the undisturbed rate of strain dyadic s is symmetric and traceless (i.e. $\mathbf{I}:\mathbf{s} = 0$), then [G.22] may be written as

$$E^+ = \mathbf{F}\cdot(\mathbf{v}^o - \mathbf{U}) + \mathbf{L}\cdot(\boldsymbol{\omega} - \boldsymbol{\Omega}) + 5\mu_o V_p \mathbf{A}:\mathbf{s}, \tag{G.25a}$$

or, equivalently, in Cartesian tensor notation,

$$E^+ = F_i(v_i^o - U_i) + L_i(\omega_i - \Omega_i) + 5\mu_o V_p A_{ij} s_{ij}. \tag{G.25b}$$

The force, torque and stresslet required in the above expression are given generally by [2.9]–[2.11]. In view of the inequalities [G.20], the material tensors appearing in [2.9]–[2.11] must therefore satisfy certain inequalities. In the context of present applications, attention will be directed only to those material constants which are relevant to axisymmetric particles.

For centrally symmetric bodies, [2.9]–[2.11] adopt the following simpler forms at the center of symmetry of the particle:

$$F_i = \mu_o {}^t\hat{K}_{ij}(v_j^o - U_j),$$
$$L_i = 6\mu_o V_p[{}^rK_{ij}(\omega_j - \Omega_j) + \tau_{ijk}s_{jk}],$$
$$A_{ij} = N_{ijk}(\omega_k - \Omega_k) + Q_{ijkl}s_{kl},$$

in which [2.12b, c] and [2.19] have been employed. Substitution of these expressions into [G.25b] and subsequent use of [2.18] yields

$$\begin{aligned} E^+ = {} & \mu_o {}^t\hat{K}_{ij}(v_i^o - U_i)(v_j^o - U_j) + 6\mu_o V_p {}^rK_{ij}(\omega_i - \Omega_i)(\omega_j - \Omega_j) \\ & + 10\mu_o V_p N_{ijk}s_{ij}(\omega_k - \Omega_k) + 5\mu_o V_p Q_{ijkl}s_{ij}s_{kl}, \end{aligned} \tag{G.26}$$

in which [2.18] has been utilized.

Since the velocity parameters $\mathbf{v}^o - \mathbf{U}$, $\boldsymbol{\omega} - \boldsymbol{\Omega}$ and s may be independently chosen, the non-negative nature of the quantity $E^+ - 2\mu_o V_p s_{mn}s_{mn}$ requires that

$${}^t\hat{K}_{ij}(v_i^o - U_i)(v_j^o - U_j) > 0 \quad \text{for } v_k^o - U_k \neq 0, \tag{G.27}$$

$${}^rK_{ij}(\omega_i - \Omega_i)(\omega_j - \Omega_j) > 0 \quad \text{for } \omega_k - \Omega_k \neq 0, \tag{G.28}$$

$$Q_{ijkl}s_{ij}s_{kl} - \tfrac{2}{5}s_{mn}s_{mn} > 0 \quad \text{for } s_{qr} \neq 0, \tag{G.29}$$

and

$$6{}^{r}K_{ij}(\omega_i - \Omega_i)(\omega_j - \Omega_j) + 10N_{ijk}s_{ij}(\omega_k - \Omega_k) + 5Q_{ijkl}s_{ij}s_{kl} - 2s_{mn}s_{mn} > 0 \quad \text{for } \omega_k - \Omega_k \neq 0 \text{ and } s_{qr} \neq 0. \qquad [G.30]$$

Inequalities [G.27] and [G.28], which apply, in fact, for bodies of any shape, require that the 3 × 3 matrices $\|{}^{t}\mathbf{K}\|$ and $\|{}^{r}\mathbf{K}\|$ be positive-definite forms. These conditions are well known (Brenner 1964b). For axisymmetric bodies they lead to the conclusion that

$${}^{t}\hat{K}_{\parallel} > 0, \quad {}^{t}\hat{K}_{\perp} > 0, \qquad [G.31a, b]$$

and

$${}^{r}K_{\parallel} > 0, \quad {}^{r}K_{\perp} > 0, \qquad [G.32a, b]$$

as follows from [2.20] and [2.21] by noting that in a system of body-fixed coordinates $O\bar{x}_i$ $(i = 1, 2, 3)$ with $O\bar{x}_3$ as the symmetry axis,

$$\|{}^{t}\mathbf{K}\| = \begin{Vmatrix} {}^{t}\hat{K}_{\perp} & 0 & 0 \\ 0 & {}^{t}\hat{K}_{\perp} & 0 \\ 0 & 0 & {}^{t}K_{\parallel} \end{Vmatrix},$$

and

$$\|{}^{r}\mathbf{K}\| = \begin{Vmatrix} {}^{r}K_{\perp} & 0 & 0 \\ 0 & {}^{r}K_{\perp} & 0 \\ 0 & 0 & {}^{r}K_{\parallel} \end{Vmatrix},$$

in which it has been noted that the unit orientational vector **e** in this body-fixed system possesses the components

$$\bar{e}_m = \delta_{m3} \quad (m = 1, 2, 3). \qquad [G.33]$$

Equations [G.31] and [G.32] are all satisfied by each of the bodies whose properties are explicitly tabulated in Section 3.

By identity,

$$s_{mn}s_{mn} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})s_{ij}s_{kl}, \qquad [G.34]$$

whence [G.29] may be written in the form

$$[Q_{ijkl} - \tfrac{1}{5}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]s_{ij}s_{kl} > 0 \quad \text{for } s_{qr} \neq 0. \qquad [G.35]$$

By means of [G.33] the Q_{ijkl} tensor for axisymmetric particles, given by [2.24], may be written in body coordinates. Inequality [G.35] therefore requires that

$$0 < [\bar{Q}_{ijkl} - \tfrac{1}{5}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\bar{s}_{ij}\bar{s}_{kl} = J + 4(Q_1 - \tfrac{1}{5})\bar{s}_{12}^2 + 4(Q_1 - \tfrac{1}{5} + Q_3)(\bar{s}_{23}^2 + \bar{s}_{31}^2), \qquad [G.36]$$

in which

$$J = [4(Q_1 - \tfrac{1}{5}) - 3Q_2]\bar{s}_{11}^2 + 2[2(Q_1 - \tfrac{1}{5}) - 3Q_2]\bar{s}_{11}\bar{s}_{22} + [4(Q_1 - \tfrac{1}{5}) - 3Q_2]\bar{s}_{22}^2$$

$$\equiv \|\bar{s}_{11}, \bar{s}_{22}\| \begin{Vmatrix} 4(Q_1 - \tfrac{1}{5}) - 3Q_2, & 2(Q_1 - \tfrac{1}{5}) - 3Q_2 \\ 2(Q_1 - \tfrac{1}{5}) - 3Q_2, & 4(Q_1 - \tfrac{1}{5}) - 3Q_2 \end{Vmatrix} \begin{Vmatrix} \bar{s}_{11} \\ \bar{s}_{22} \end{Vmatrix}. \qquad [G.37]$$

In arriving at these relations, the identities

$$\bar{s}_{ij} = \bar{s}_{ji}, \qquad [G.38]$$

and

$$\bar{s}_{11} + \bar{s}_{22} + \bar{s}_{33} = 0 \qquad [G.39]$$

have been utilized, the latter to eliminate $\bar{s}_{33}$. Since the five strain components in the body-fixed system, $\bar{s}_{11}, \bar{s}_{22}, \bar{s}_{12}, \bar{s}_{23}$ and $\bar{s}_{31}$, may be chosen independently of one another, the inequality [G.36] requires that

$$Q_1 > 1/5, \qquad [G.40]$$

$$Q_1 + Q_3 > 1/5, \qquad [G.41]$$

and $J > 0$ for $\bar{s}_{ij} \neq 0$. The latter requires that the 2×2 matrix in [G.37] be a positive-definite form, i.e.

$$4(Q_1 - \tfrac{1}{5}) - 3Q_2 > 0,$$

and, upon forming the determinant of the matrix,

$$[4(Q_1 - \tfrac{1}{5}) - 3Q_2]^2 > [2(Q_1 - \tfrac{1}{5}) - 3Q_2]^2,$$

which are, respectively, equivalent to

$$\tfrac{4}{3}(Q_1 - \tfrac{1}{5}) > Q_2,$$

and, upon squaring the terms enclosed in square brackets,

$$Q_1 - \tfrac{1}{5} > Q_2. \qquad [G.42]$$

In view of [G.40] we have that $\tfrac{4}{3}(Q_1 - \tfrac{1}{5}) > Q_1 - \tfrac{1}{5}$, whence it follows that the equation immediately preceding [G.42] will automatically be satisfied if [G.42] is itself satisfied.

For an axially symmetric body we find with use of [2.21], [2.22] and [2.24], along with [G.33], that, in body coordinates, the inequality [G.30] requires that

$$0 < 6\mu_o V_p[{}^r\!K_\perp(\bar{W}_1^2 + \bar{W}_2^2) + {}^r\!K_\parallel \bar{W}_3^2] + 20\mu_o V_p N(\bar{W}_2 \bar{s}_{31} - \bar{W}_1 \bar{s}_{23})$$
$$+ 5\mu_o V_p[J + 4(Q_1 - \tfrac{1}{5})\bar{s}_{12}^2 + 4(Q_1 - \tfrac{1}{5} + Q_3)(\bar{s}_{23}^2 + \bar{s}_{31}^2)],$$

in which we have put $\bar{W}_i = \bar{\omega}_i - \bar{\Omega}_i$ and utilized [G.38] and [G.39]. The quantity J is as defined in [G.37]. Since $\bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{s}_{11}, \bar{s}_{22}, \bar{s}_{12}, \bar{s}_{23}$ and $\bar{s}_{31}$ may be chosen independently, the above inequality will be satisfied if [G.32], [G.40], [G.41] and [G.42] are satisfied, and if also

$$J' \overset{\text{def.}}{=} 6[{}^r\!K_\perp \bar{W}_1^2 - \tfrac{10}{3} N \bar{W}_1 \bar{s}_{23} + \tfrac{10}{3}(Q_1 - \tfrac{1}{5} + Q_3)\bar{s}_{23}^2] > 0,$$

and

$$J'' \overset{\text{def.}}{=} 6[{}^r\!K_\perp \bar{W}_2^2 + \tfrac{10}{3} N \bar{W}_2 \bar{s}_{31} + \tfrac{10}{3}(Q_1 - \tfrac{1}{5} + Q_3)\bar{s}_{31}^2] > 0.$$

As in the case of [G.37], satisfaction of these two inequalities requires that the two 2×2 matrices

$$\left\| \begin{matrix} {}^rK_\perp, & -\frac{5}{3}N \\ -\frac{5}{3}N, & \frac{10}{3}(Q_1 - \frac{1}{5} + Q_3) \end{matrix} \right\| \quad \text{and} \quad \left\| \begin{matrix} {}^rK_\perp, & \frac{5}{3}N \\ \frac{5}{3}N, & \frac{10}{3}(Q_1 - \frac{1}{5} + Q_3) \end{matrix} \right\|$$

both be positive definite. This will be the case if ${}^rK_\perp > 0$, $\frac{10}{3}(Q_1 - \frac{1}{5} + Q_3) > 0$, and if

$$\tfrac{10}{3}K_\perp(Q_1 - \tfrac{1}{5} + Q_3) > (\pm\tfrac{5}{3}N)^2.$$

The first two of these inequalities are already contained in [G.32b] and [G.41], while the latter requires that

$$Q_1 + Q_3 - \tfrac{1}{5} > \left(\frac{5}{3}\frac{N}{{}^rK_\perp}\right)\frac{N}{2}.$$

From [2.28b] the term in parentheses is the dimensionless angular velocity parameter B. Therefore, from the definition of Q_3^o in [2.36] the latter inequality requires that

$$Q_1 + Q_3^o > 1/5. \tag{G.43}$$

Equations [G.40]-[G.43] constitute the inequalities imposed on the Q material constants by the requirement that the additional energy dissipation rate satisfy [G.19]. Each of these four relations is satisfied by all of the bodies whose properties are tabulated in Section 3.

Energy dissipation in a dilute suspension of force-free and couple-free Brownian particles. Upon putting $\mathbf{F} = 0$ and $\mathbf{L} = 0$ in [G.25a] there is obtained

$$E^+ = 5\mu_o V_p \mathbf{A}:\mathbf{s}. \tag{G.44}$$

According to [3.4] and [2.17c],

$$\mathbf{A} = \mathbf{s}:\mathbf{Q}^o \tag{G.45}$$

in the present circumstances. Inasmuch as $\mathbf{Q}^o \equiv \mathbf{Q}^o(\mathbf{e})$ is a function of the orientation $\mathbf{e}$ of the particle, this makes $E^+ \equiv E^+(\mathbf{e})$. Accordingly, [G.44] gives the additional energy dissipation rate resulting from the presence of a force- and couple-free axisymmetric particle suspended in a homogeneous shearing flow and possessing an instantaneous orientation $\mathbf{e}$ relative to, say, the principal axes of the undisturbed shear $\mathbf{s}$.

The additional mechanical energy dissipation rate (per unit time) per unit superficial volume of suspension, specific to particles of orientation $\mathbf{e}$, is therefore $5\mu_o\phi\mathbf{A}:\mathbf{s}$. Consequently, the additional dissipation rate D^+ per unit volume due to particles of all orientations is

$$D^+ = 5\mu_o\phi \oint \mathbf{A}:\mathbf{s}f(\mathbf{e})\, d^2\mathbf{e}.$$

In consequence of [4.4a], [4.18], [4.21] and [4.6a], this may be written as

$$D^+ = 5\mu_o\phi G^2\langle\mathbf{A}'\rangle:\hat{\mathbf{S}},$$

correct to the first order in ϕ. The rate of dissipation D^∞ per unit volume of suspension due to the fluid alone being subjected to a mean shear $\mathbf{S}$ is

$$D^\infty = 2\mu_o \mathbf{S}:\mathbf{S} \equiv 2\mu_o G^2 \hat{\mathbf{S}}:\hat{\mathbf{S}}. \tag{G.46}$$

Hence, the total dissipation rate D per unit volume,

$$D = D^\infty + D^+, \tag{G.47}$$

is given by the expression

$$D = 2\mu_o G^2[\hat{\mathbf{S}} + \tfrac{5}{2}\phi\langle \mathbf{A}'\rangle]:\hat{\mathbf{S}},$$

to the first order in ϕ. Comparison with [4.23] shows that this may be written as

$$D = \mathbf{T}:\mathbf{S}, \tag{G.48}$$

where $\mathbf{T}$ is the mean deviatoric stress in the suspension. Thus, in the absence of external forces and couples, the dissipation rate per unit volume of suspension is equal to the product of the mean deviatoric stress with the mean rate of strain. This is a very satisfying result from a continuum mechanical point of view, since it is precisely what one would have anticipated (for a symmetric state of stress).

Equation [G.48] constitutes the generalization to the case where rotary Brownian motion is sensible of a similar result due to Goddard & Miller (1967), Frankel & Acrivos (1970), and Batchelor (1970).

From the inequality [G.19] in conjunction with [4.4a] we find that to the first order in ϕ,

$$D^+ \geq 2\mu_o \phi \mathbf{S}:\mathbf{S}.$$

With use of [G.46], [G.47] and [G.48] this furnishes the following lower bound on the dissipation rate:

$$D = \mathbf{T}:\mathbf{S} \geq 2\mu_o(1 + \phi)\mathbf{S}:\mathbf{S} \geq 0, \tag{G.49}$$

in which the equality sign applies only when $\mathbf{S} = 0$.*

APPENDIX H

Evaluation of the normalization constant K defined in [10.60]

Substitute the expression for $\Delta_\perp$ derived from [10.31] into [10.60] and utilize the displayed equations following [10.61]. In this manner one obtains

$$K = \oint \exp(-\mathbf{e}\cdot\mathbf{C}\cdot\mathbf{e})\, d^2\mathbf{e} \tag{H.1}$$

* Note from [2.9] to [2.11], in conjunction with the fact that the suspended particles are both force-free and couple-free, that the conditions [G.11] will all be satisfied if $\mathbf{s} = 0$.

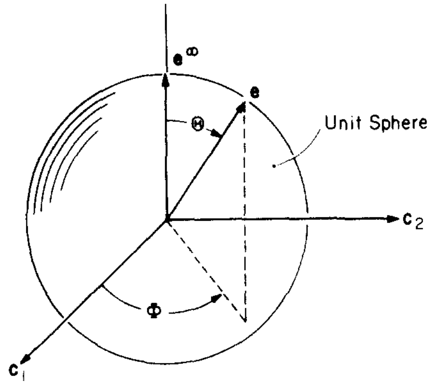

Figure 23. Definition sketch of the spherical polar angles Θ and Φ.

to terms of dominant order in the small parameter D_r. Integration is over the unit sphere. Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be mutually perpendicular unit vectors lying along the principal axes of the symmetric planar dyadic $\mathbf{C}$, and denote by C_1 and C_2 its principal values, both of which are positive. Thus, we may write

$$\mathbf{C} = \mathbf{c}_1\mathbf{c}_1 C_1 + \mathbf{c}_2\mathbf{c}_2 C_2. \qquad [\text{H.2}]$$

(In the interests of generality we will not assume that $\mathbf{A}$ is symmetric, so that $\mathbf{c}_1$ and $\mathbf{c}_2$ are not necessarily identical to the unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, appearing in [10.63].) As in figure 23, the triad $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{e}^\infty)$ constitutes a mutually perpendicular system of unit vectors.

Let (Θ, Φ) be the spherical polar angles defined in figure 23, with Θ the polar angle measured from $\mathbf{e}^\infty$ and the Φ the azimuthal angle measured from $\mathbf{c}_1$. In this system we may write that

$$\mathbf{e} = \mathbf{c}_1 \sin\Theta\cos\Phi + \mathbf{c}_2 \sin\Theta\sin\Phi + \mathbf{e}^\infty \cos\Phi. \qquad [\text{H.3}]$$

Since $f(\mathbf{e}) = f(-\mathbf{e})$, equation [H.1] may be integrated over the unit hemisphere $0 \le \Theta \le \pi/2$, rather than over the complete unit sphere $0 \le \Theta \le \pi$, and the value of the resulting integral doubled to obtain K. Hence,

$$K = 2\int_{\Phi=0}^{2\pi}\int_{\Theta=0}^{\pi/2} \exp[-C(\Phi)\sin^2\Theta]\sin\Theta\, d\Theta\, d\Phi, \qquad [\text{H.4}]$$

wherein

$$C(\Phi) = C_1\cos^2\Phi + C_2\sin^2\Phi. \qquad [\text{H.5}]$$

Since $\sin^2\Theta = 1 - \cos^2\Theta$ and $\sin\Theta\, d\Theta = -d(\cos\Theta)$, it is possible to effect the Θ integration analytically in terms of the error function. However, we will content ourselves with performing an asymptotic integration, consistent with the asymptotic nature of [10.58] itself.

Inasmuch as $C = O(D_r^{-1})$, in the limit of weak Brownian motion the integrand of [H.4] is nonzero only in the immediate vicinity of the pole, $\Theta = 0$. Consequently, in this limit we may utilize the approximation $\sin\Theta \sim \Theta$. Hence, [H.4] may be written as

$$K = 2\int_{\Phi=0}^{2\pi} F(\Phi)\,\mathrm{d}\Phi, \qquad \text{[H.6]}$$

where

$$F(\Phi) = \int_{\Theta=0}^{\infty} \exp(-C\Theta^2)\Theta\,\mathrm{d}\Theta. \qquad \text{[H.7]}$$

Insofar as dominant terms are concerned, it is immaterial that we have replaced by upper limit of integration, $\Theta = \pi/2$, by $\Theta = \infty$. This is a consequence of the fact that the integrand of [H.7] is everywhere vanishingly small in the limit as $D_r \to 0$, except in the immediate proximity of the pole, $\Theta = 0$. Asymptotically, the resultant error is negligible.*

Straightforward integration of [H.7] yields

$$F(\Phi) = \frac{1}{2C(\Phi)}. \qquad \text{[H.8]}$$

Therefore,

$$K = \int_0^{2\pi} \frac{\mathrm{d}\Phi}{C(\Phi)} \equiv 2\int_0^{\pi} \frac{\mathrm{d}\Phi}{C(\Phi)} = 2[I(C_1, C_2) + I(C_2, C_1)],$$

in which

$$I(C_1, C_2) = \int_0^{\pi/2} \frac{\mathrm{d}\Phi}{C_1\cos^2\Phi + C_2\sin^2\Phi} = \frac{\pi}{2(C_1C_2)^{1/2}}. \qquad \text{[H.9]}$$

Since, in general, $\det \mathbf{C} = C_1C_2$, this yields

$$K = 2\pi(\det \mathbf{C})^{-1/2}. \qquad \text{[H.10]}$$

The second moment $\langle \mathbf{ee} \rangle$ required in the rheological calculations can be calculated by writing

$$\langle \mathbf{ee} \rangle = \int_{S_{\mathrm{I}}} \mathbf{ee}f\,\mathrm{d}^2\mathbf{e} + \int_{S_{\mathrm{II}}} \mathbf{ee}f\,\mathrm{d}^2\mathbf{e},$$

* This asymptotic procedure can be made more rigorous (Brenner 1970) by introducing an "inner" stretched variable $\tilde{\Theta} = \Theta/\sqrt{D_r}$ in place of Θ. This new variable possesses the property of being of $O(1)$ in the proximity of the pole for small D_r. Indeed, such a procedure provides an alternate and more systematic asymptotic technique for solving the basic orientational diffusion equation,

$$\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}(f\dot{\Theta}\sin\Theta) + \frac{\partial}{\partial\Phi}(f\dot{\Phi}) = D_r\frac{1}{\sin\Theta}\left[\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial f}{\partial\Theta}\right) + \frac{1}{\sin\Theta}\frac{\partial^2 f}{\partial\Phi^2}\right],$$

where (Θ, Φ) are the appropriate spherical polar angles measured relative to the stable terminal orientation $\mathbf{e}^{\infty}$. This procedure can be employed to obtain higher order terms in the expansion.

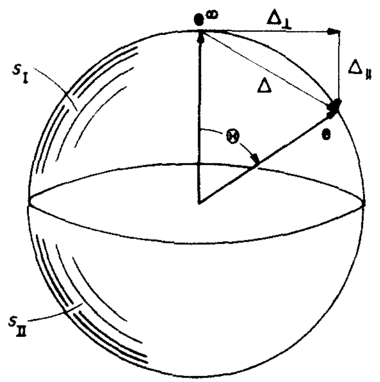

Figure 24. Definition sketch for the components of the perturbation vector $\boldsymbol{\Delta}$ relative to the direction $\mathbf{e}^{\infty}$ of the terminal orientation of an axisymmetric particle.

in which $S_{\rm I}$ refers to the upper hemisphere $(0 \le \Theta \le \pi/2)$ in figure 24, and $S_{\rm II}$ to the lower hemisphere $(\pi/2 \le \Theta \le \pi)$. Since both $f(\mathbf{e})$ and $\mathbf{ee}$ are even functions of $\mathbf{e}$ (cf. [10.62]) the above may be written as

$$\langle \mathbf{ee} \rangle = 2 \int_{S_{\rm I}} \mathbf{ee} f \, d^2\mathbf{e}. \qquad [\text{H.11}]$$

On $S_{\rm I}$ we have exactly that

$$\mathbf{e} = \mathbf{e}^{\infty} + \boldsymbol{\Delta}_{\parallel} + \boldsymbol{\Delta}_{\perp} \quad \text{on } S_{\rm I}, \qquad [\text{H.12}]$$

where $\boldsymbol{\Delta}_{\parallel}$ and $\boldsymbol{\Delta}_{\perp}$ are the vectors depicted in figure 24. Thus, since $\Delta_{\parallel} = O(\Delta_{\perp}^2)$ we have that on $S_{\rm I}$,

$$\mathbf{ee} = \mathbf{e}^{\infty}\mathbf{e}^{\infty} + \mathbf{e}^{\infty}\boldsymbol{\Delta}_{\parallel} + \boldsymbol{\Delta}_{\parallel}\mathbf{e}^{\infty} + \mathbf{e}^{\infty}\boldsymbol{\Delta}_{\perp} + (\mathbf{e}^{\infty}\boldsymbol{\Delta}_{\perp})^{\dagger} + \boldsymbol{\Delta}_{\perp}\boldsymbol{\Delta}_{\perp} + O(\Delta^3). \qquad [\text{H.13}]$$

By definition, $\boldsymbol{\Delta}_{\parallel}$ is colinear with $\mathbf{e}^{\infty}$. We may therefore write

$$\boldsymbol{\Delta}_{\parallel} = \kappa \mathbf{e}^{\infty}.$$

To determine κ, dot multiply [H.12] by $\mathbf{e}^{\infty}$, thereby obtaining

$$\cos\Theta = \mathbf{e} \cdot \mathbf{e}^{\infty} = 1 + \kappa,$$

whence

$$\boldsymbol{\Delta}_{\parallel} = -\mathbf{e}^{\infty}(1 - \cos\Theta) \quad \text{on } S_{\rm I}. \qquad [\text{H.14}]$$

Since $\boldsymbol{\Delta}_{\perp}$ is perpendicular to $\mathbf{e}^{\infty}$, we have by identity that

$$\boldsymbol{\Delta}_{\perp} = (\mathbf{I} - \mathbf{e}^{\infty}\mathbf{e}^{\infty}) \cdot \boldsymbol{\Delta}_{\perp},$$

whence, upon utilizing [H.12],

$$\boldsymbol{\Delta}_{\perp} = (\mathbf{I} - \mathbf{e}^{\infty}\mathbf{e}^{\infty}) \cdot \mathbf{e}. \qquad [\text{H.15}]$$

Collecting results gives

$$\langle \mathbf{ee} \rangle = 2\mathbf{e}^{\infty}\mathbf{e}^{\infty} \int_{S_I} f\, d^2\mathbf{e} - 4\mathbf{e}^{\infty}\mathbf{e}^{\infty} \int_{S_I} (1 - \cos\Theta) f\, d^2\mathbf{e}$$
$$+ 2(\boldsymbol{\chi} + \boldsymbol{\chi}^{\dagger}) + 2 \int_{S_I} \boldsymbol{\Delta}_{\perp}\boldsymbol{\Delta}_{\perp} f\, d^2\mathbf{e} + O(\Delta^3), \quad \text{[H.16]}$$

in which $\boldsymbol{\chi}$ is the dyadic

$$\boldsymbol{\chi} = \mathbf{e}^{\infty}(\mathbf{I} - \mathbf{e}^{\infty}\mathbf{e}^{\infty}) \cdot \int_{S_I} \mathbf{e} f\, d^2\mathbf{e}. \quad \text{[H.17]}$$

From [H.3] and [10.62] we have that

$$f(\Theta, \Phi) = K^{-1} \exp[-(C_1 \cos^2\Phi + C_2 \sin^2\Phi)\sin^2\Theta]. \quad \text{[H.18]}$$

Therefore,

$$\int_{S_I} \mathbf{e} f\, d^2\mathbf{e} = \mathbf{c}_1 \int_{\Theta=0}^{\pi/2} d\Theta \sin^2\Theta \int_{\Phi=0}^{2\pi} \cos\Phi f(\Theta, \Phi)\, d\Phi$$
$$+ \mathbf{c}_2 \int_{\Theta=0}^{\pi/2} d\Theta \sin^2\Theta \int_{\Phi=0}^{2\pi} \sin\Phi f(\Theta, \Phi)\, d\Phi$$
$$+ \mathbf{e}^{\infty} \int_{\Theta=0}^{\pi/2} \int_{\Phi=0}^{2\pi} \sin\Theta \cos\Theta\, f(\Theta, \Phi)\, d\Theta\, d\Phi.$$

The Φ integrations in the first two integrals above may be subdivided into integrals from $\Phi = 0$ to π and $\Phi = \pi$ to 2π. In this manner it is readily shown that

$$\int_{\Phi=0}^{2\pi} \begin{Bmatrix} \cos\Phi \\ \sin\Phi \end{Bmatrix} f(\Theta, \Phi)\, d\Phi = 0,$$

whence the first two integrals vanish. Thus, $\int_{S_I} \mathbf{e} f\, d^2\mathbf{e}$ possesses, at most, a component in the $\mathbf{e}^{\infty}$ direction. But this component is annihilated by the operator $\mathbf{I} - \mathbf{e}^{\infty}\mathbf{e}^{\infty}$ in [H.17]. Consequently,

$$\boldsymbol{\chi} = 0. \quad \text{[H.19]}$$

Moreover, since f is an even function of $\mathbf{e}$, we have in [H.16], from the normalization condition imposed on f, that

$$\int_{S_I} f\, d^2\mathbf{e} = \tfrac{1}{2}. \quad \text{[H.20]}$$

Likewise, since f is an even function of $\boldsymbol{\Delta}_{\perp}$ (cf. [10.58]), the last integral in [H.16] can be expressed as half the comparable value over the entire unit sphere. In this manner one obtains

$$\langle \mathbf{ee} \rangle = \mathbf{e}^{\infty}\mathbf{e}^{\infty}(1 - 4\gamma) + \langle \boldsymbol{\Delta}_{\perp}\boldsymbol{\Delta}_{\perp} \rangle + O(\Delta^3), \quad \text{[H.21]}$$

in which

$$\langle \mathbf{\Delta}_\perp \mathbf{\Delta}_\perp \rangle = \oint \mathbf{\Delta}_\perp \mathbf{\Delta}_\perp f(\mathbf{\Delta}_\perp)\, d^2\mathbf{e}, \qquad [H.22]$$

and

$$\gamma = \int_{\Phi=0}^{2\pi} \int_{\Theta=0}^{\pi} (1 - \cos\Theta) f(\Theta, \Phi) \sin\Theta\, d\Theta\, d\Phi. \qquad [H.23]$$

The integral [H.22] can be evaluated by noting from [10.60] that

$$\langle \mathbf{\Delta}_\perp \mathbf{\Delta}_\perp \rangle = -\frac{1}{K}\frac{\partial K}{\partial \mathbf{C}} \equiv -\frac{\partial \ln K}{\partial \mathbf{C}},$$

or, using [H.2],

$$\langle \mathbf{\Delta}_\perp \mathbf{\Delta}_\perp \rangle = -\left(\mathbf{c}_1\mathbf{c}_1 \frac{\partial}{\partial C_1} + \mathbf{c}_2\mathbf{c}_2 \frac{\partial}{\partial C_2}\right) \ln K. \qquad [H.24]$$

From [H.19] this yields

$$\langle \mathbf{\Delta}_\perp \mathbf{\Delta}_\perp \rangle = \frac{1}{2}\left(\frac{\mathbf{c}_1\mathbf{c}_1}{C_1} + \frac{\mathbf{c}_2\mathbf{c}_2}{C_2}\right) \equiv \tfrac{1}{2}\mathbf{C}^{-1}, \qquad [H.25]$$

which applies irrespective of whether or not $\mathbf{A}$ is symmetrical.

The integral [H.23] can be asymptotically evaluated by observing that f is nonzero only in the immediate vicinity of the pole $\Theta = 0$. Thus, expansion of the trigonometric functions in Θ for small Θ, with use of [H.18] yields

$$\gamma = \tfrac{1}{2}K^{-1} \int_{\Phi=0}^{2\pi} \int_{\Theta=0}^{\infty} \exp[-(C_1\cos^2\Phi + C_2\sin^2\Phi)\Theta^2]\Theta^3\, d\Theta. \qquad [H.26]$$

However, from [H.4] and [H.5] we have asymptotically that

$$K = 2\int_{\Phi=0}^{2\pi} \int_{\Theta=0}^{\infty} \exp[-(C_1\cos^2\Phi + C_2\sin^2\Phi)\Theta^2]\Theta\, d\Theta.$$

Since $\cos^2\Phi + \sin^2\Phi = 1$, it readily follows from this that the integral appearing in [H.26] is

$$-\frac{1}{2}\left(\frac{\partial K}{\partial C_1} + \frac{\partial K}{\partial C_2}\right).$$

Consequently,

$$\gamma = -\frac{1}{4}\left(\frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_2}\right) \ln K. \qquad [H.27]$$

Comparison with [H.24] shows that

$$\gamma = \tfrac{1}{4}\,\mathrm{tr}\,\langle \mathbf{\Delta}_\perp \mathbf{\Delta}_\perp \rangle, \qquad [H.28]$$

and thus from [H.25],

$$\gamma = \tfrac{1}{8}\,\mathrm{tr}\,(\mathbf{C}^{-1}) \equiv \frac{1}{8}\left(\frac{1}{C_1} + \frac{1}{C_2}\right). \qquad [H.29]$$

Substitution of [H.25] and [H.29] into [H.21] thereby yields

$$\langle \mathbf{ee} \rangle = [1 - \tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{C}^{-1})]\mathbf{e}^{\infty}\mathbf{e}^{\infty} + \tfrac{1}{2}\mathbf{C}^{-1} \qquad [H.30]$$

to terms of dominant order in D_r^{-1}. Note that since $\mathrm{tr}\,(\mathbf{e}^{\infty}\mathbf{e}^{\infty}) = \mathbf{e}^{\infty}\cdot\mathbf{e}^{\infty} = 1$ this result correctly reduces to the form $\mathrm{tr}\,\langle \mathbf{ee} \rangle = 1$, a necessary consequence of the fact that $\mathbf{e}\cdot\mathbf{e} = 1$.

The values of the normalization constant [H.10] and the second moment [H.30] differ from the values obtained by Hinch (1971), namely

$$K_{\text{Hinch}} = \frac{2}{\pi}\det \mathbf{C},$$

and

$$\langle \mathbf{ee} \rangle_{\text{Hinch}} = \mathbf{e}^{\infty}\mathbf{e}^{\infty} + \mathbf{C}^{-1}.$$

That our values for these two parameters, rather than Hinch's, are correct can be confirmed by reference to a particular example, where the exact orientational distribution function is known for any degree of intensity of the Brownian movement, by then passing to the limiting case where $D_r \to 0$.

With this in mind, consider the axisymmetric extensional flow field [5.1] for the case where

$$BG > 0. \qquad [H.31]$$

With use of [5.3] and [5.4], equation [10.16] gives

$$\mathbf{H} = (1/2)BG(3\mathbf{i}_3\mathbf{i}_3 - \mathbf{I}). \qquad [H.32]$$

As can be demonstrated from [10.17] and [10.18] (Brenner 1972a, 1972c), in the absence of rotary Brownian motion an axisymmetric body for which [H.31] holds adopts a stable terminal orientation given by

$$\mathbf{e}^{\infty} = \mathbf{i}_3. \qquad [H.33]$$

As thereupon follows from [10.19], the eigenvalue corresponding to this orientation is

$$h = BG. \qquad [H.34]$$

Equation [10.34] thus yields

$$\mathbf{A} = -(3/2)BG(\mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2). \qquad [H.35]$$

The eigenvalues of this symmetric dyadic are $A_1 = A_2 = -3BG/2$. (Both being negative, this confirms the stability of the terminal orientation [H.33]).

From [10.59] we find that

$$\mathbf{C} = \mathbf{i}_1\mathbf{i}_1 C_1 + \mathbf{i}_2\mathbf{i}_2 C_2, \qquad [H.36]$$

in which

$$C_1 = C_2 = 3BG/4D_r = \xi^2, \quad \text{say}, \tag{H.37}$$

where ξ is the dimensionless parameter defined in [5.10], λ there being BG/D_r (cf. [4.16]). Substitution of [H.36], [H.37] and [5.5] into [10.62] now yields

$$f(\theta) = \frac{1}{2\pi}\xi^2 \exp(-\xi^2 \sin^2\theta) \tag{H.38}$$

for the value of the distribution function in the limit where $\xi \gg 1$. From [5.6] and [5.7a], upon putting $\cos^2\theta = 1 - \sin^2\theta$, the exact value of the distribution function for this case is

$$f(\theta) = (K')^{-1} \exp(-\xi^2 \sin^2\theta), \tag{H.39}$$

in which

$$K' = 4\pi\xi^{-1}D(\xi), \tag{H.40}$$

where $D(\xi)$ is Dawson's integral, defined in [5.8]. For $\xi \gg 1$ we have the asymptotic expansion*

$$D(\xi) = \frac{1}{2\xi}\left[1 + \frac{1}{2\xi^2} + \frac{1\cdot 3}{2^2\xi^4} + \frac{1\cdot 3\cdot 5}{2^3\xi^6} + \cdots\right], \tag{H.41}$$

whence we obtain

$$K' = 2\pi/\xi^2 \quad \text{for } \xi \gg 1. \tag{H.42}$$

Substitution into [H.39] yields a result identical to [H.28]. This calculation confirms the correctness of the normalization constant [H.10].

Similar confirmation of the second moment expression [H.30] is furnished by the present example. From [H.36], [H.37] and [H.33] we obtain

$$\langle \mathbf{ee} \rangle = \mathbf{i}_3\mathbf{i}_3 + (\mathbf{I} - 3\mathbf{i}_3\mathbf{i}_3)\frac{1}{2\xi^2}. \tag{H.43}$$

The exact value for arbitrary ξ is given by [5.11] and [5.12a]. With use of [H.41] we find that

$$F(\xi) = 1 - \frac{1}{\xi^2} + O\left(\frac{1}{\xi^4}\right) \quad \text{for } \xi \gg 1.$$

Substitution into [5.11] then yields a result identical to [H.43], thereby confirming the correctness of [H.30].

* This semi-convergent expansion can be obtained by methods similar to those utilized in obtaining the asymptotic expansion of the complementary error function (Carslaw & Jaeger 1959). That this expansion yields accurate results can be confirmed by reference to the numerical values of Dawson's integral tabulated by Abramowitz & Stegun (1968).

Sommaire—Des résultats explicites sont présentés pour les propriétés rhéologiques complètes de suspensions diluées de particules Browniennes axisymmétriques rigides possédant une symétrie avantarrière lorsqu'elles sont suspendues dans un liquide Newtonien soumis à un écoulement de cisaillement tri-dimensionel général, soit équilibré soit déséquilibré. Il est montré que ces propriétés rhéologiques peuvent être exprimées en fonction de cinq constantes fondamentales matérielles (à l'exclusion de la viscosité du solvant) et qui ne dépendent que de la grandeur et de la forme des particules suspendues. Des expressions sont présentées pour ces constantes scalaires pour un nombre de solides de révolution, y compris les sphéroïdes, les haltères de rapport d'aspect arbitraire et les corps longs et minces. Celles-ci sont utilisées pour calculer les propriétés rhéologiques d'une variété d'écoulements de cisaillement différents, y compris les écoulements uniaxiaux et biaxiaux extensionels, les écoulements de cisaillement simples et ceux bi-dimensionels généraux. Il est démontré que les propriétés rhéologiques applicables à un écoulement de cisaillement général bi-dimensionel peuvent être déduites immédiatement de celles d'un écoulement de cisaillement simple. Cette observation accroit grandement l'utilité d'une grande partie de la littérature sur l'écoulement Couette, surtout les calculs numériques étendus de Scheraga *et al.* (1951, 1955).

La communalité de nombreux résultats disparates, répandus et diffusés dans des publications antérieures est soulignée et présentée d'un point de vue unifié hydrodynamique.

Auszug—Es werden bestimmte Ergebnisse für die vollständigen Rheologieeigenschaften verdünnter Suspension von starren, achsensymmetrischen Brown'schen Teilchen dargestellt, welche längslaufende Symmetrie besitzen, wenn sie in einer Newton'schen Flüssigkeit unter Einfluß eines allgemeinen dreidimensionalen Scherflusses, gleichmäßig oder ungleichmäßig, aufgeschlämmt werden. Es wird gezeigt, daß diese Rheologieeigenschaften in Form von fünf elementaren Materialkonstanten ausgedrückt werden können (ausschließlich der Viskosität des Lösungsmittels), welche nur von den Größen und Formen der Suspensionsteilchen abhängen. Es werden Ausdrücke für diese skalaren Konstanten für eine Anzahl fester Rotationskörper, einschließlich Sphäroiden, Hanteln von beliebigem Längenverhältnis und langen, schlanken Körpern dargestellt. Diese werden zur Berechnung rheologischer Eigenschaften für eine Auswahl verschiedener Scherflüsse verwandt, einschließlich einachsiger und zweiachsiger Streckungsflüsse, einfacher Scherflüsse und allgemeiner, zweidimensionaler Scherflüsse. Es wird dargestellt, daß die, zu einem allgemeinen, zweidimensionalen Scherflß gehörigen, rheologischen Eigenschaften sofort von den Eigenschaften für einen einfachen Scherfluß abgeleitet werden können. Diese Beobachtung vergrößert die Nützlichkeit von vielen der Schriften vor der Conette Fließliteratur, besonders der ausführlichen zahlenmäßigen Berechnungen von Scheraga *et al.* (1951, 1955).

Es wird die Allgemeinheit vieler unvereinbarer Ergebnisse betont, die in früheren Veröffentlichungen verstreut und verteilt waren, und diese werden von einem einheitlichen hydrodynamischen Standpunkt aus dargestellt.

Резюме—Представляются исчерпывающие результаты на полные реологические характеристики разбавленных суспензий жестких, осесимметричных Броунианoвых частиц имеющих симметрию во всю длину, когда они взвешены в ньютоновской жидкости, подверженной общему установившемуся или неустановившемуся трехмерному течению с поперечным градиентом скорости. Демонстрируется, что эти реологические характеристики можно выразить в пяти основных материальных константах, исключая вязкость растворителя, зависящей только от размера и формы взвешенных частиц. Даются выражения этих скалярных констант на теиа вращения сфероидов, гантелей произвольного вида и длинных тонких тел. Они применяются для вычисления реологических характеристик различных типов течений с поперечным градиентом скорости, включая одноосные и двухосные течения, простые течения с поперечным градиентом скорости, и общие двухмерные течения с поперечным градиентом скорости. Нашли, что реологические характеристики присущие общему двухмерному течению с поперечным градиентом скорости можно вывести из характеристик простого течения с поперечным градиентом скорости. Это определение на много повышает ценность прежней литературы о течении Куэтта, особенно, численные расчеты Шерага и др. (1951, 1955 г).

В итоге, этой работой придается особое значение многим отличающимся друг от друга результатам и вопрос реологических характеристик представляется с унифицированной гидродинамической точки зрения.